WINDOWING

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Based on [Berg], [Mount]

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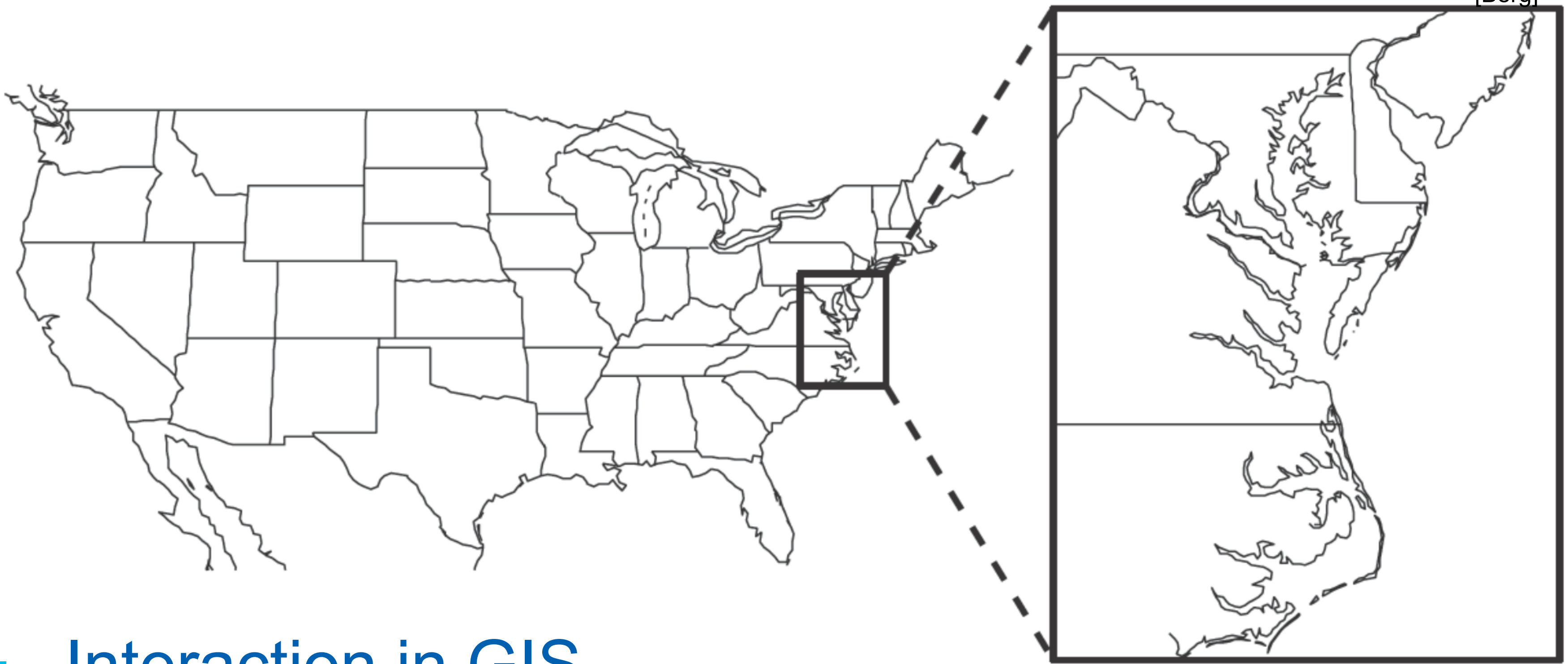
Talk overview

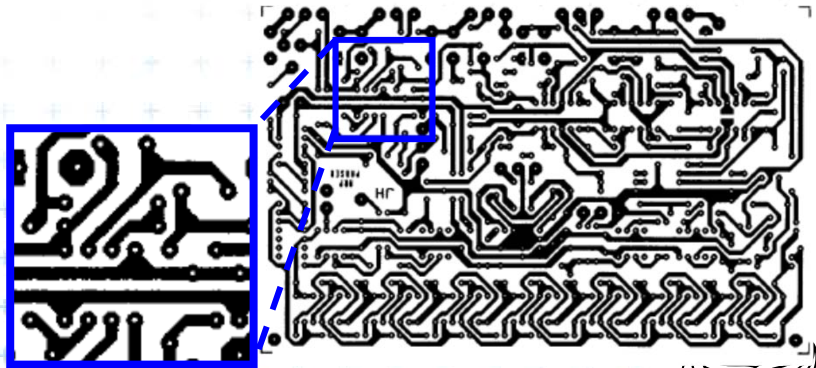
- Windowing
- Windowing of axis parallel line segments (interval tree - IT)
  - Line stabbing (*interval tree* with sorted lists)
  - Line segment stabbing (*IT* with *range trees*)
  - Line segment stabbing (*IT* with *priority search trees*)
- Windowing of line segments in general position
  - *segment tree*

Windowing queries - examples

[Berg]



- Interaction in GIS
  - Select subset by outlining
  - Zoom in and re-center
- Circuit board inspection,…



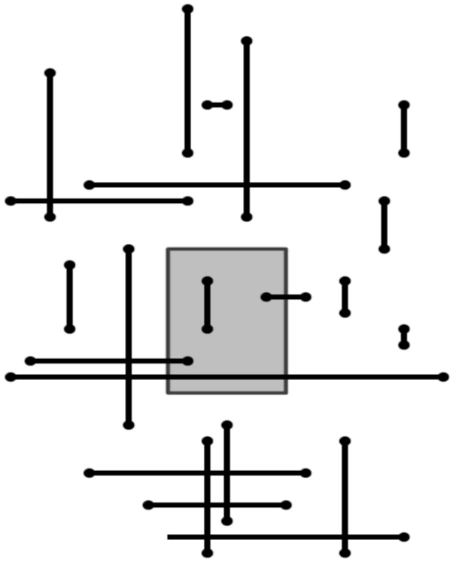
[Vakken]

Windowing versus range queries

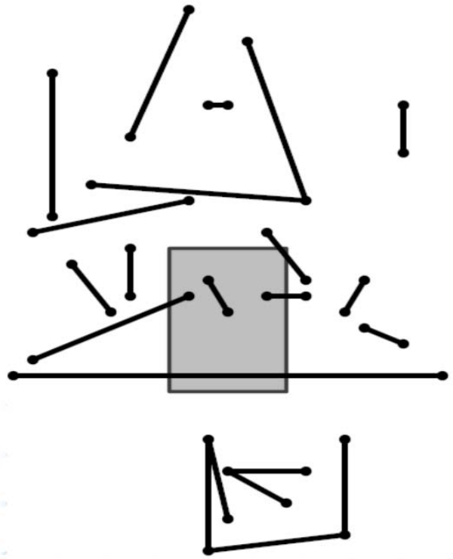
- Range queries (range trees in Lecture 03)
  - Points
  - Often in higher dimensions
- Windowing queries
  - Line segments, curves, …
  - Usually in low dimension (2D, 3D)

Windowing queries

- Preprocess the data into a data structure
  - so that the ones intersected by the query rectangle can be reported efficiently
- Two cases



Axis parallel line segments



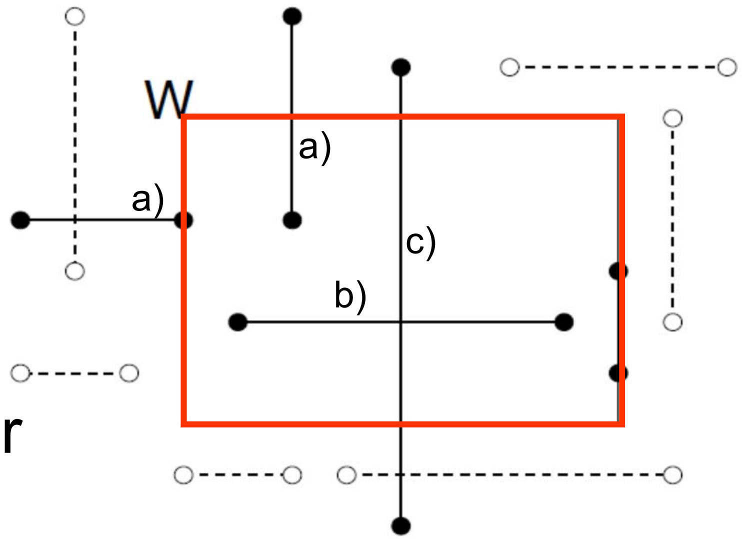
Arbitrary line segments (non-crossing)

[Vakken]

Windowing of axis parallel line segments

Window query

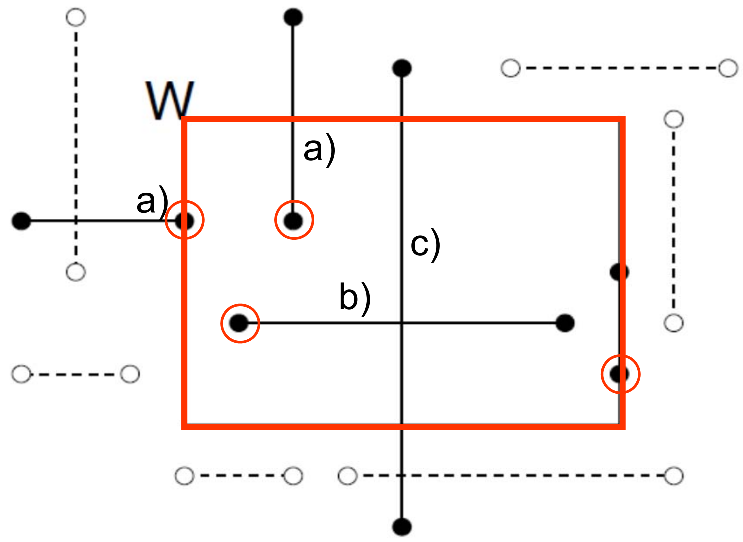
- Given
  - a set of orthogonal line segments *S* (preprocessed),
  - and orthogonal query rectangle $W = [\, x : x' \,] \times [\, y : y' \,]$
- Count or report all the line segments of *S* that intersect *W*
- Such segments have
  - a) 1 endpoint in
  - b) 2 end points in – Included
  - c) no end point in – Cross over



[Mount]

Line segments with 1 or 2 points inside

a) 1 point inside
   – Use a range tree (Lesson 3)
   – $O(n \log n)$ storage
   – $O(\log^2 n + k)$ query time or
   – $O(\log n + k)$ with fractional cascading

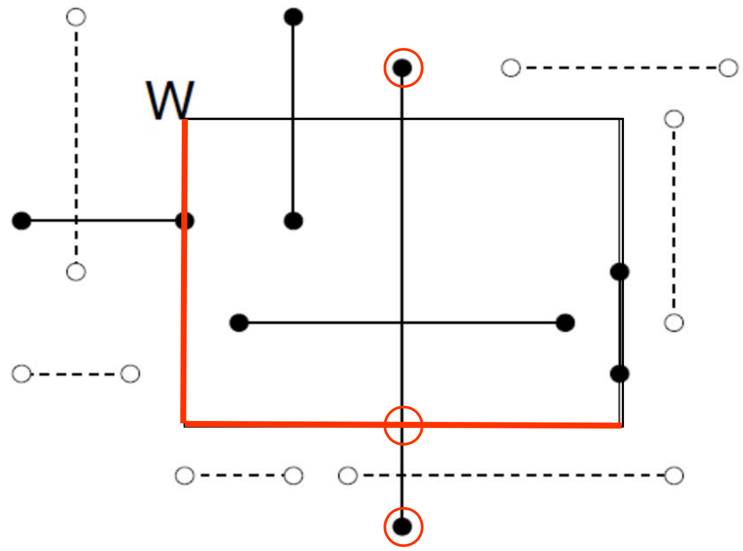


b) 2 points inside – as a) 1 point inside
   – Avoid reporting twice
   1. Mark segment when reported (clear after the query)
   2. When end point found, check the other end-point. Report only the leftmost or bottom endpoint

Line segments that cross over the window

c) No points inside

- not detected using a range tree
- Cross the boundary twice
  or
  contain one boundary edge
- It is enough to detect segments intersected by the left and bottom boundary edges (not having end point inside)
- For left boundary: Report the segments intersecting *vertical query line segment* (B)
- Let's discuss vertical query *line* first (A)
- Bottom boundary is rotated 90°



A: Segment intersected by vertical line

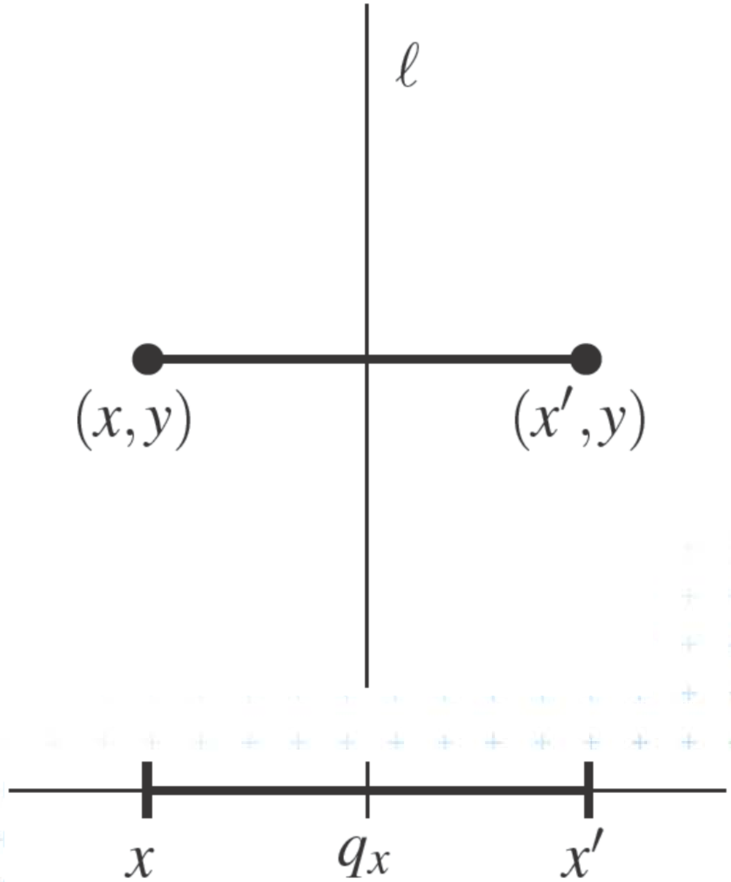
- Query line $\ell := (x=q_x)$

  Report the segments stabbed by a vertical line

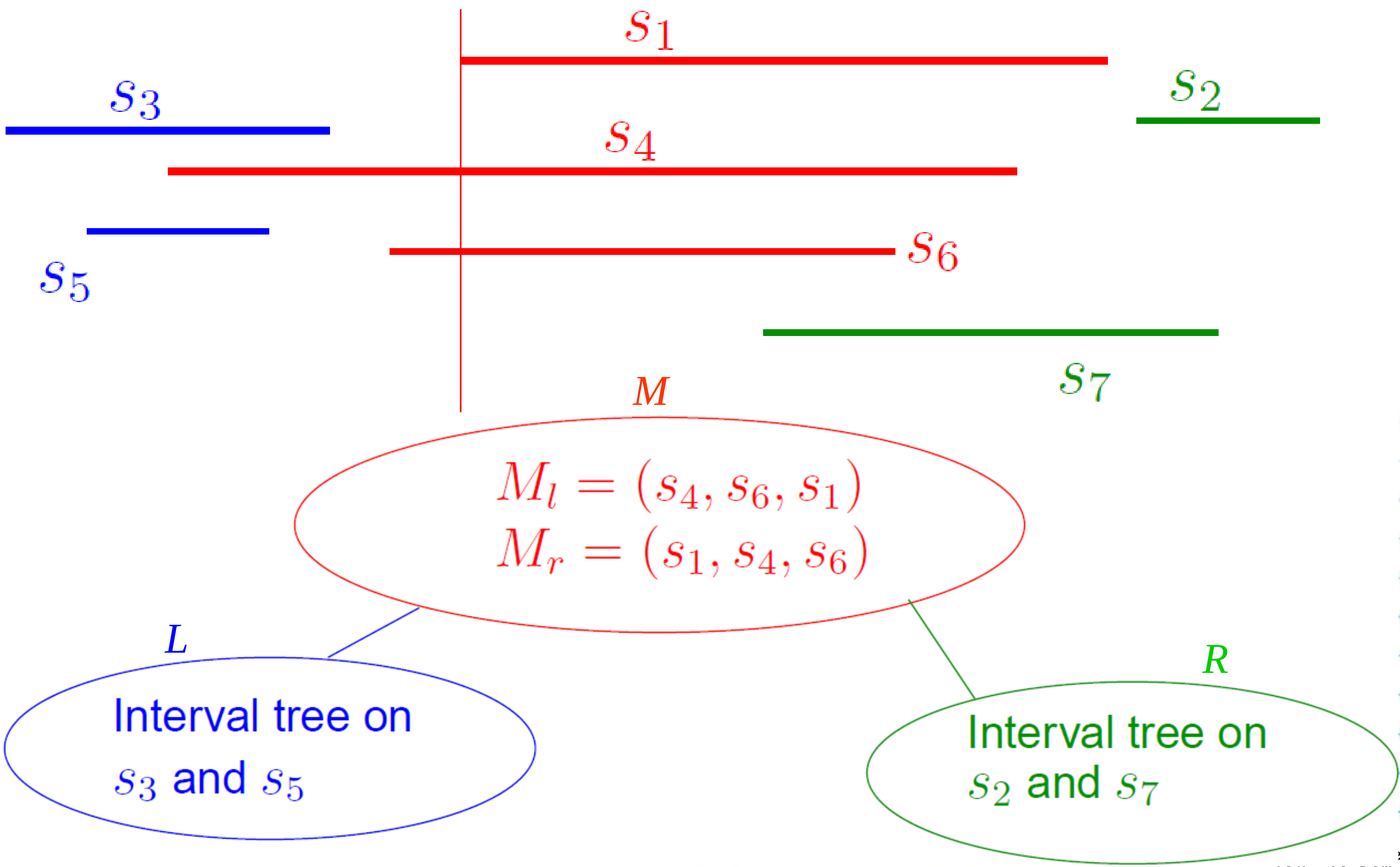
  = 1 dimensional problem

  (ignore y coordinate)

=> Report the interval containing query point q_x

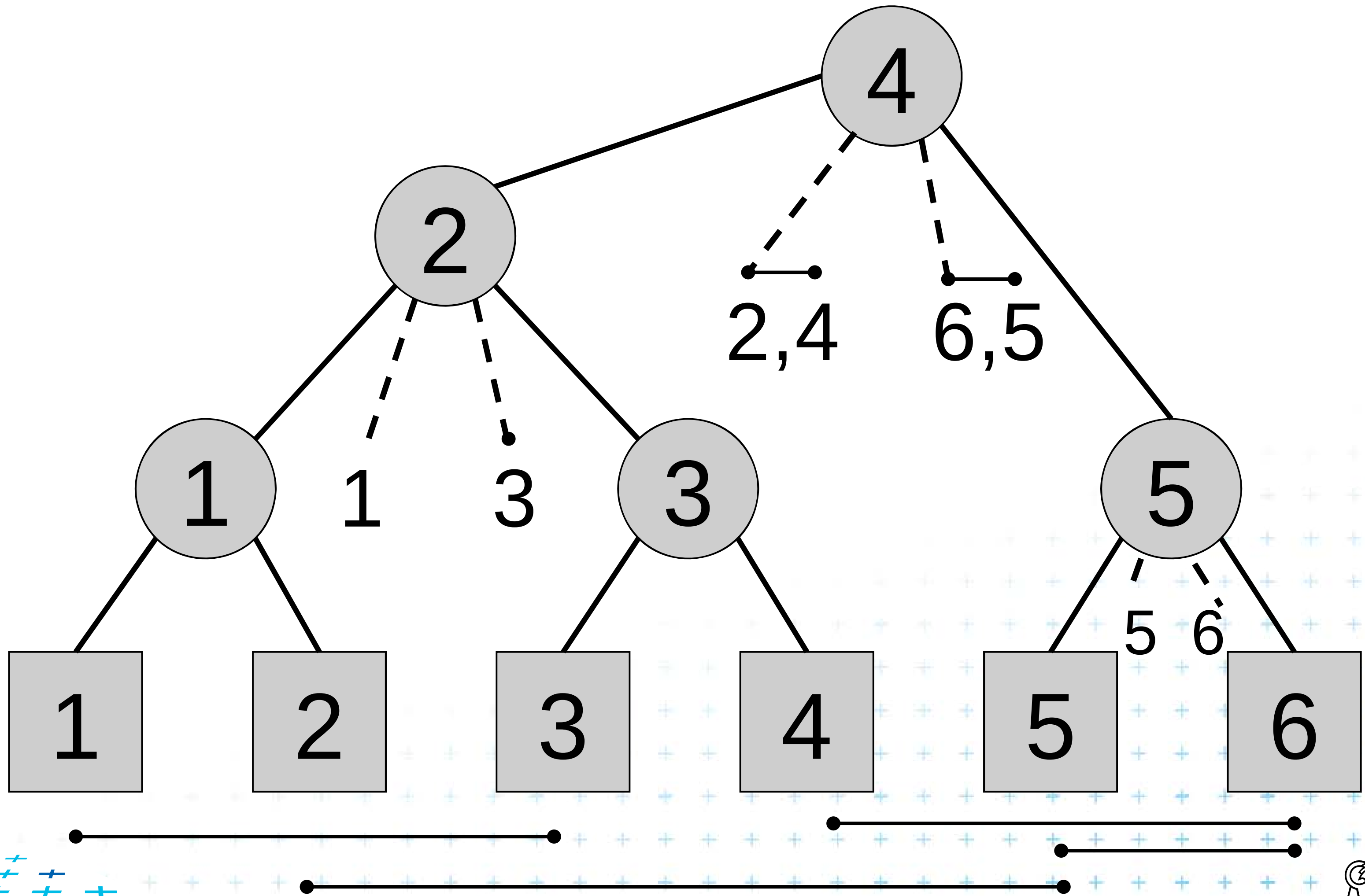


Interval tree principle



[Vigneron]

Static interval tree [Edelsbrunner80]

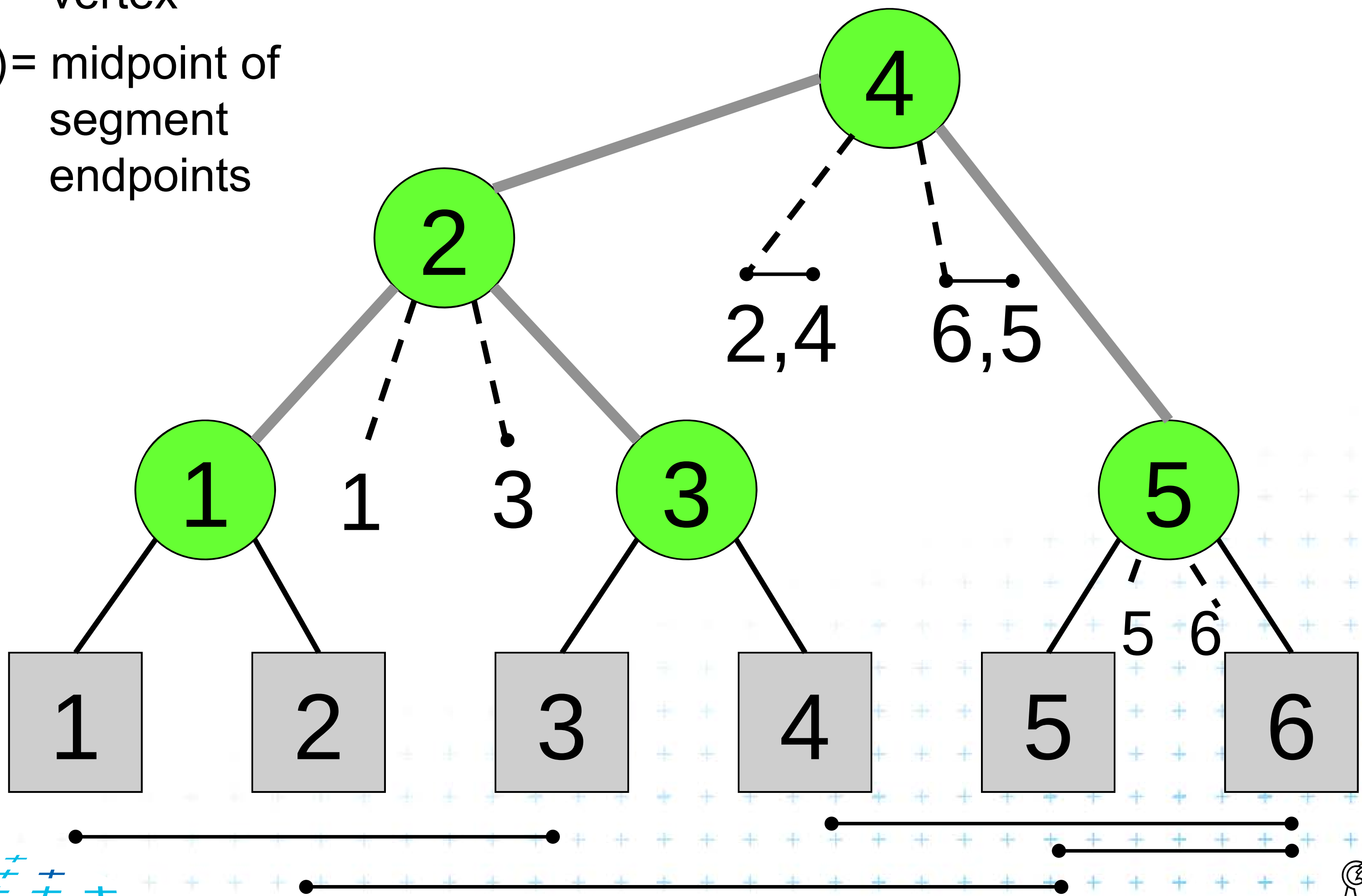


[Kukral]

Primary structure – static tree for endpoints

v = vertex

d(v)= midpoint of segment endpoints

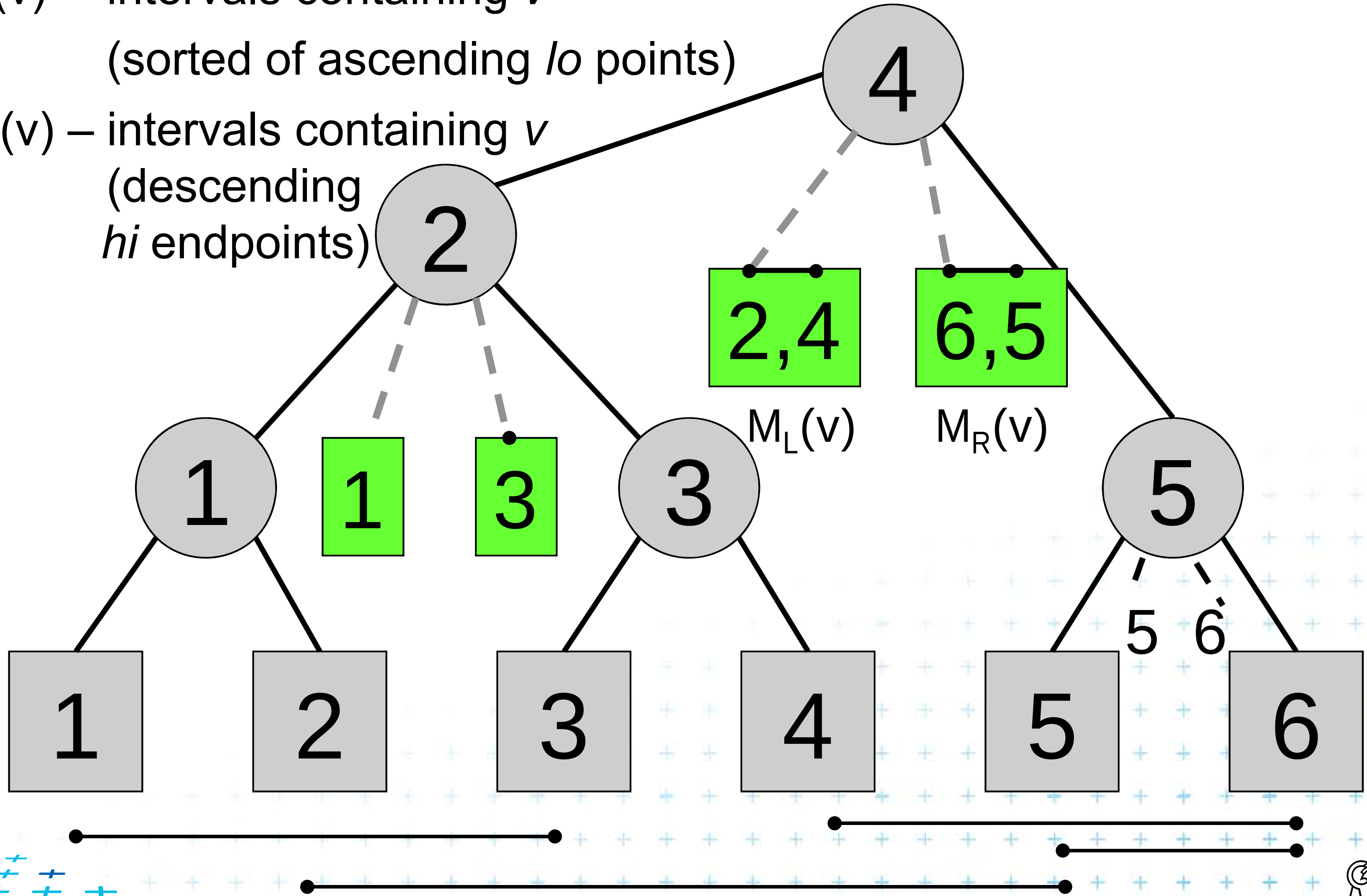


[Kukral]

Secondary lists – sorted segments in M

ML(v) – intervals containing *v*
(sorted of ascending *lo* points)

MR(v) – intervals containing *v*
(descending *hi* endpoints)



[Kukral]

Interval tree construction

ConstructIntervalTree(*S*)

Input: Set S of intervals on the real line

Output: The root of an interval tree for *S*

```
if (|S| == 0) return null                                // no more
else
    xMed = median endpoint of intervals in S             // median endpoint
    L = { [xlo, xhi] in S | xhi < xMed }                 // left of median
    R = { [xlo, xhi] in S | xlo > xMed }                 // right of median
    M = { [xlo, xhi] in S | xlo <= xMed <= xhi }         // contains median
    ML = sort M in increasing order of xlo               // sort M
    MR = sort M in decreasing order of xhi
    t = new IntTreeNode(xMed, ML, MR)                    // this node
    t.left  = ConstructIntervalTree(L)                   // left subtree
    t.right = ConstructIntervalTree(R)                   // right subtree
    return t
```

[Mount]

Line stabbing query for an interval tree

```
Stab( t, xq)
Input:    IntTreeNode t, Scalar xq
Output:  prints the intersected intervals
if (t == null) return                                // fell out of tree
if (xq < t.xMed)                                     // left of median?
    for (i = 0; i < t.ML.length; i++)                // traverse ML
            if (t.ML[i].lo <= xq) print(t.ML[i])     // ..report if in range
            else break                               // ..else done
    stab(t.left, xq)                                 // recurse on left
else  // (xq ≥ t.xMed)                               // right of or equal to median
    for (i = 0; i < t.MR.length; i++) {              // traverse MR
            if (t.MR[i].hi ≥ xq) print(t.MR[i])      // ..report if in range
            else break                               // ..else done
    stab(t.right, xq)                                // recurse on right
```

Note: Small inefficiency for xq == t.xMed – recurse on right

[Mount]

Complexity of line stabbing via interval tree

- Construction - $O(n \log n)$ time
  - Each step divides at maximum into two halves or less (minus elements of M) => tree height $O(\log n)$
  - If presorted the endpoints in three lists L,R,M then median in O(1) and copy to new L,R,M in $O(n)$]
- Vertical line stabbing query - $O(k + \log n)$ time
  - One node processed in O(1 + k’), k’=reported intervals
  - v visited nodes in $O(v + k)$, k=total reported intervals
  - v = tree height = $O(\log n)$
- Storage - $O(n)$
  - Tree has $O(n)$ nodes, each segment stored twice (two endpoints)

A: Segment intersected by vertical line - 1D

- Query line $\ell := (x = q_x)$

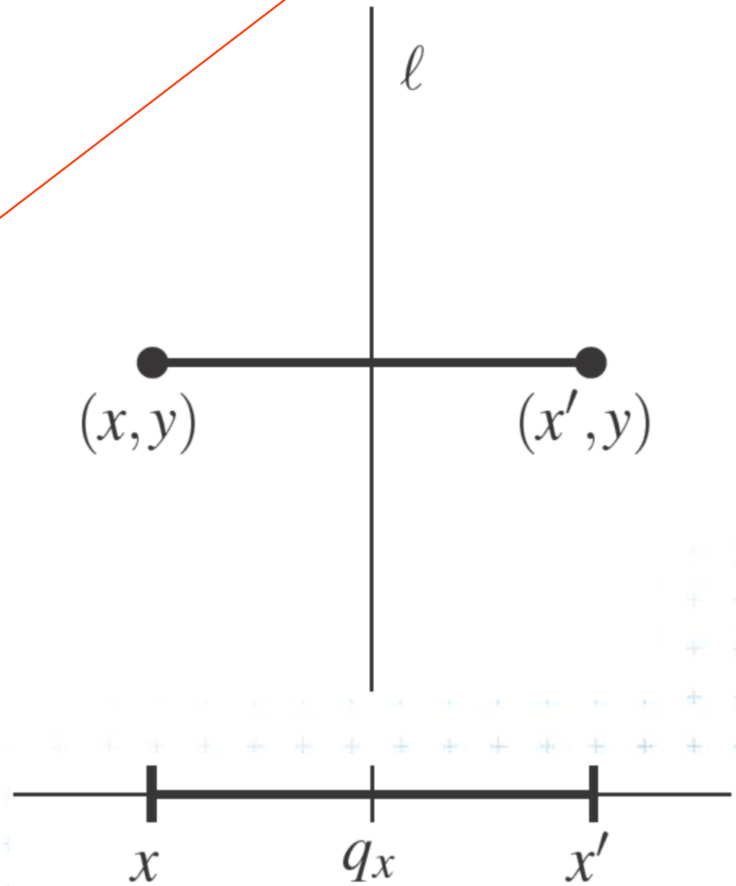
  Report the segments stabbed by a vertical line

  = 1 dimensional problem

  (ignore y coordinate)

=> Report the interval containing query point q_x

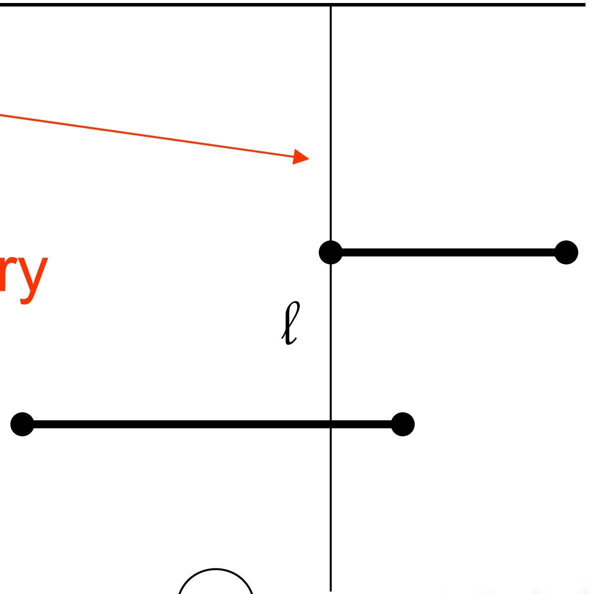
DS: Interval tree



[Berg]

A: Segment intersected by vertical line - 2D

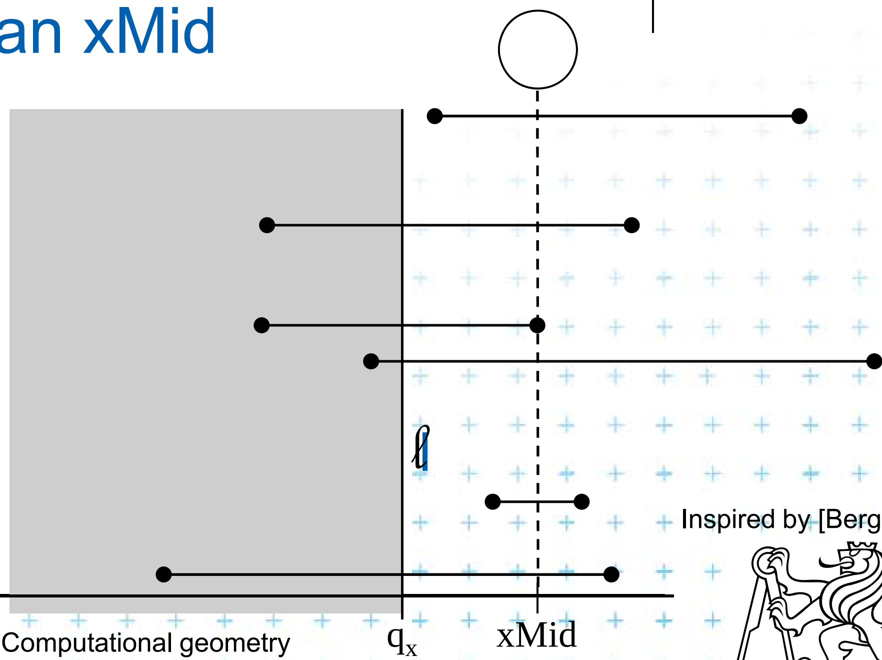
- Query line $\ell := q_x \times [-\infty : \infty]$



- Horizontal segment of M stabs the query line ℓ **iff** its left endpoint lies in halph-space

$$(-\infty : q_x] \times [-\infty : \infty]$$

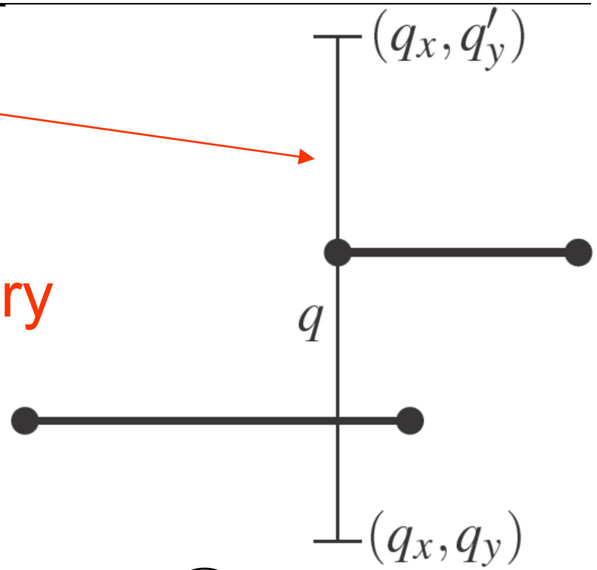
- In IT node with stored median xMid report all segments from M
  - whose left point lies in $(-\infty : q_x]$
    if ℓ lies left from xMid
  - whose right point lies in $(q_x : +\infty]$
    if ℓ lies right from xMid



Inspired by [Berg]

B: Segment intersected by vertical line segment

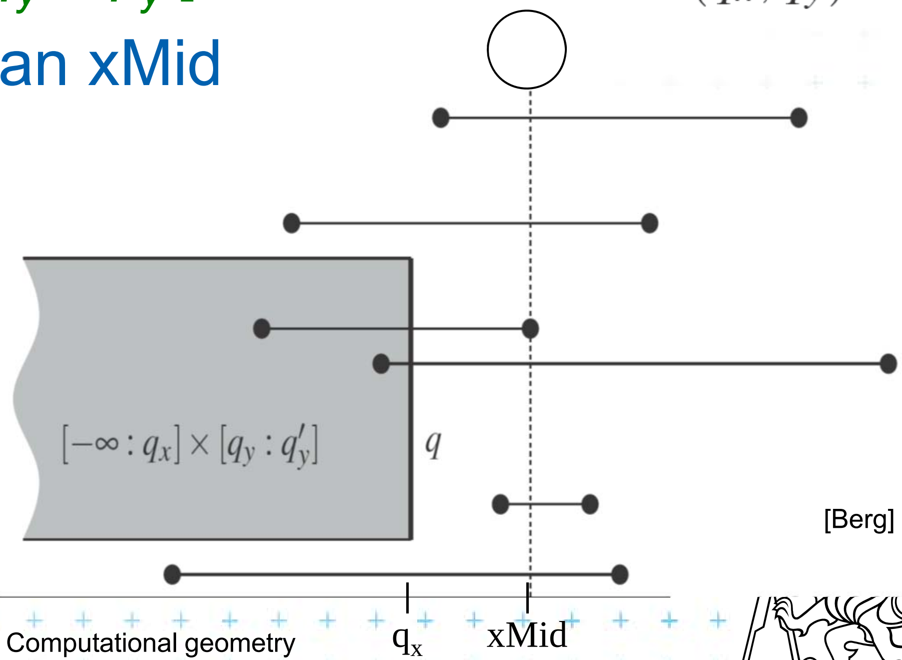
- Query segment $q := q_x \times [q_y : q'_y]$



- Horizontal segment of M stabs the query segment q **iff** its left endpoint lies in semi-infinite rectangular region

  $$(-\infty : q_x] \times [q_y ; q'_y]$$

- In IT node with stored median xMid report all segments
  - whose left point lies in $(-\infty : q_x] \times [q_y ; q'_y]$ if q lies left from xMid
  - whose right point lies in $(q_x : +\infty] \times [q_y ; q'_y]$ if q lies right from xMid

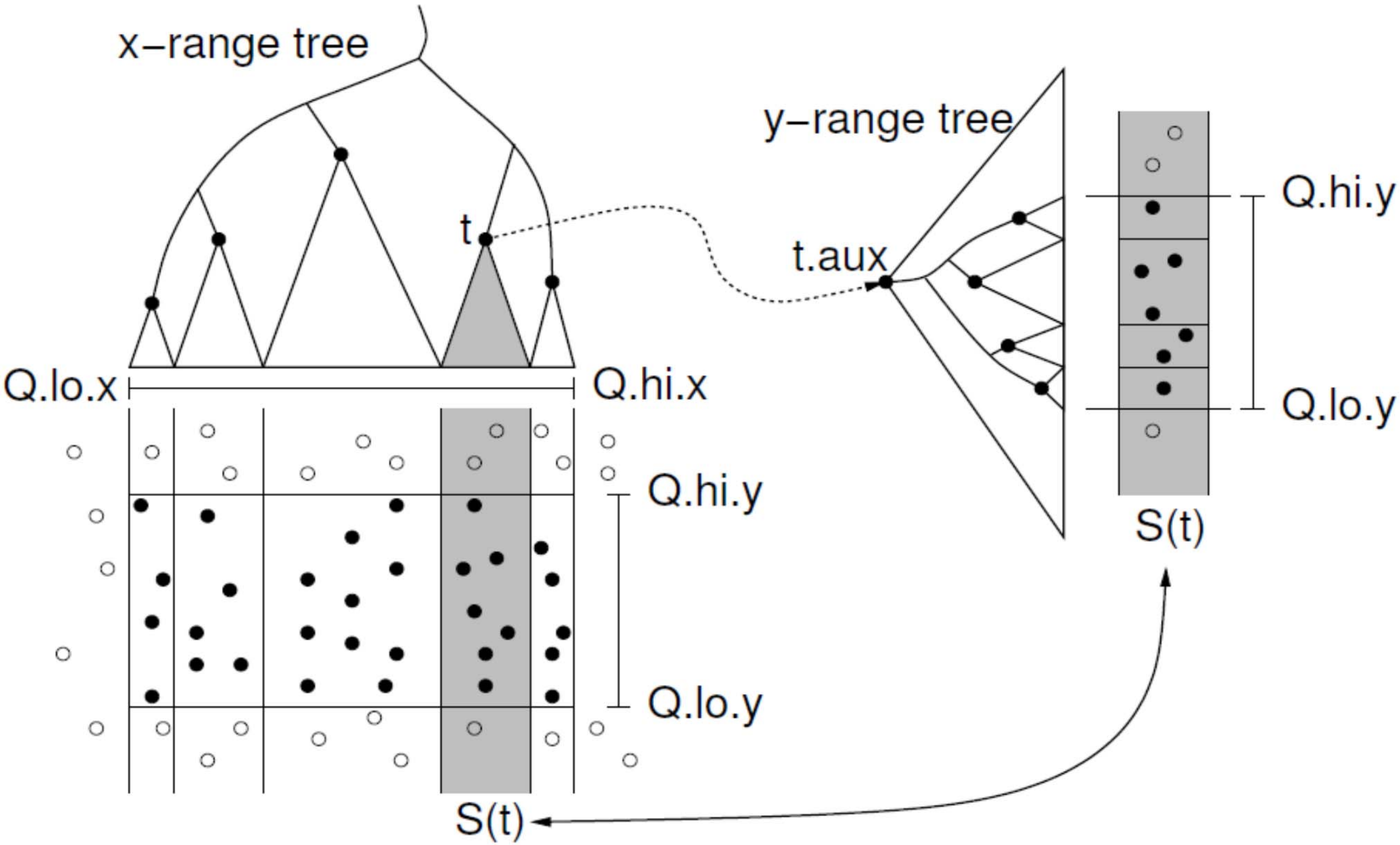


[Berg]

Data structure for endpoints

- Storage of ML and MR
  - Sorted lists not enough for line segments
  - Use two range trees
- Instead O(n) sequential search in ML and MR perform O(log n) search
  in range tree with fractional cascading

2D range tree (without fractional casc. - see more in Lecture 3)



Complexity of line segment stabbing

- Construction - $O(n \log n)$ time
  - Each step divides at maximum into two halves L,R or less (minus elements of M) => tree height $O(\log n)$
  - If the range trees are efficiently build in $O(n)$
- Vertical line segment stab. q. - $O(k + \log^2 n)$ time
  - One node processed in $O(\log n + k')$, k'=reported inter.
  - v visited nodes in $O(v \log n + k)$, k=total reported inter.
  - v = tree height = $O(\log n)$
  - $O(k + \log^2 n)$ time - range tree with fractional cascading
  - $O(k + \log^3 n)$ time - range tree without fractional casc.
- Storage - $O(n \log n)$
  - Dominated by the range trees

Priority search trees [McCreight85]

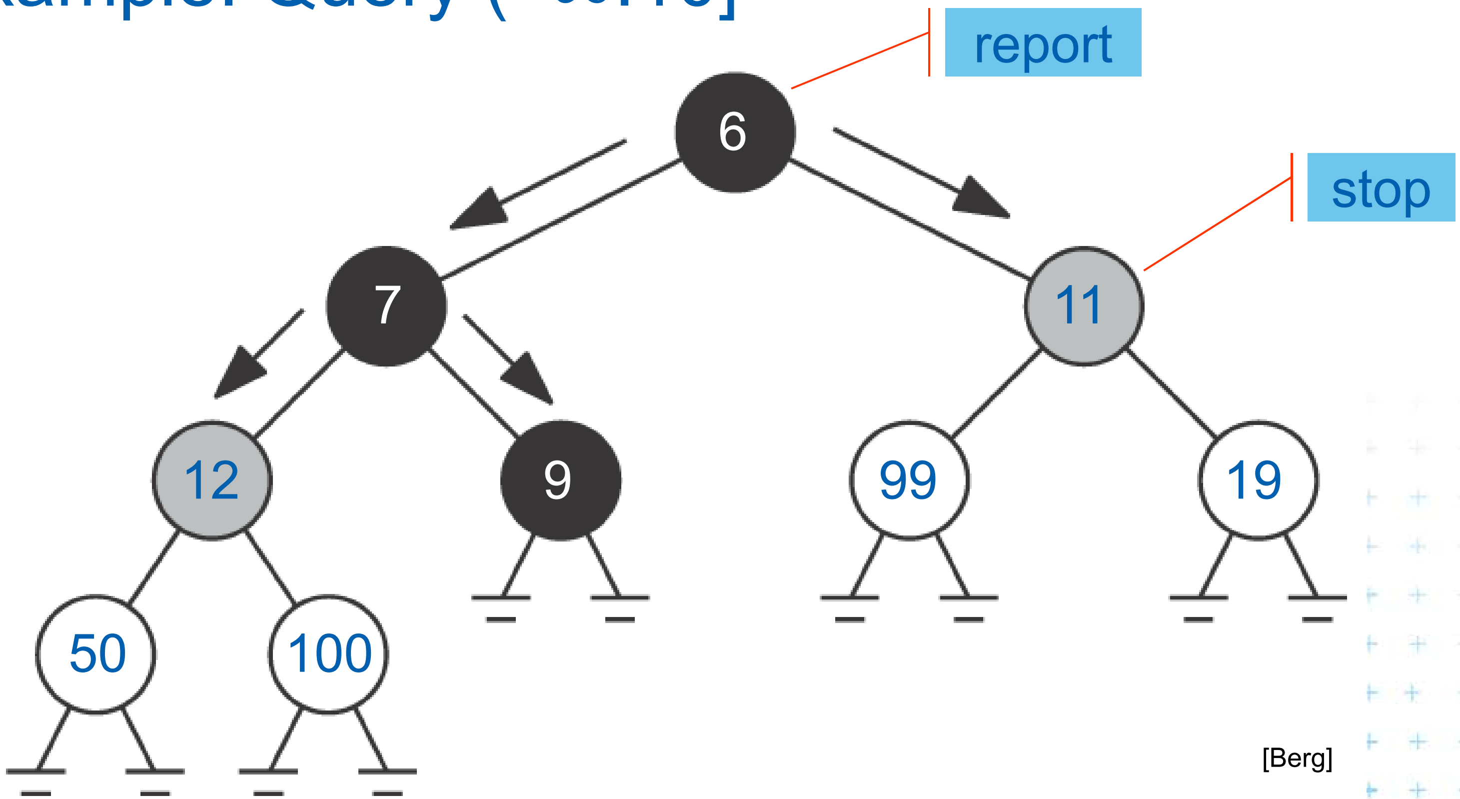
- Priority search trees – in case c) on slide 8
  - Exploit the fact that query rectangle in range queries is unbounded
  - Can be used as secondary data structures for both left and right endpoints (ML and MR) of segments (intervals) in nodes of interval tree
  - Improve the storage to O(n) for horizontal segment intersection with window edge (Range tree has O(n log n))
- For cases a) and b) - O(n log n) remains
  - we need range trees for windowing segment endpoints

Rectangular range queries variants

- Let $P = \{ p_1, p_2, \ldots, p_n \}$ is set of points in plane
- Goal: rectangular range queries of the form $(-\infty : q_x] \times [q_y ; q'_y]$
- In 1D: search for nodes v with $v_x \in (-\infty : q_x]$
  - range tree — $O(\log n + k)$ time
  - ordered list — $O(1 + k)$ time (start in the leftmost, stop on v with $v_x > q_x$)
  - use heap — $O(1 + k)$ time (traverse all children, stop when $v_x > q_x$)
- In 2D – use heap for points with $x \in (-\infty : q_x]$ + integrate information about y-coordinate

Heap for 1D unbounded range queries

- Traverse all children, stop when $v_x > q_x$
- Example: Query $(-\infty:10]$



[Berg]

Priority search tree (PST)

- Heap in 2D can incorporate info about both x,y
  - BST on y-coordinate (horizontal slabs) ~ range tree
  - Heap on x-coordinate (minimum x from slab along x)
- If P is empty, PST is empty leaf
- else
  - p_{min} = point with smallest x-coordinate in P
  - y_{med} = y-coord. median of points $P \setminus \{p_{min}\}$
  - $P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med}\}$
  - $P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med}\}$
- Point p_{min} and scalar y_{med} are stored in the root
- The left subtree is PST of P_{below}
- The right subtree is PST of P_{above}

Priority search tree definition

PrioritySearchTree(P)

Input: set P of points in plane

Output: priority search tree T

if $P=\emptyset$ then PST is an empty leaf

else

p_{min} = point with smallest x-coordinate in P

y_{med} = y-coord. median of points $P \setminus \{p_{min}\}$

Split points $P \setminus \{p_{min}\}$ into two subsets – according to y_{med}

$P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med} \}$

$P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med} \}$

T = newTreeNode()

$T.p = p_{min}$ // point [x, y]

$T.y = y_{mid}$ // skalar

$T.left$ = PrioritySearchTree(P_{below})

$T.rigft$ = PrioritySearchTree(P_{above})

O(n log n) , but O(n) if presorted on y-coordinate and bottom up

Notation in alg:

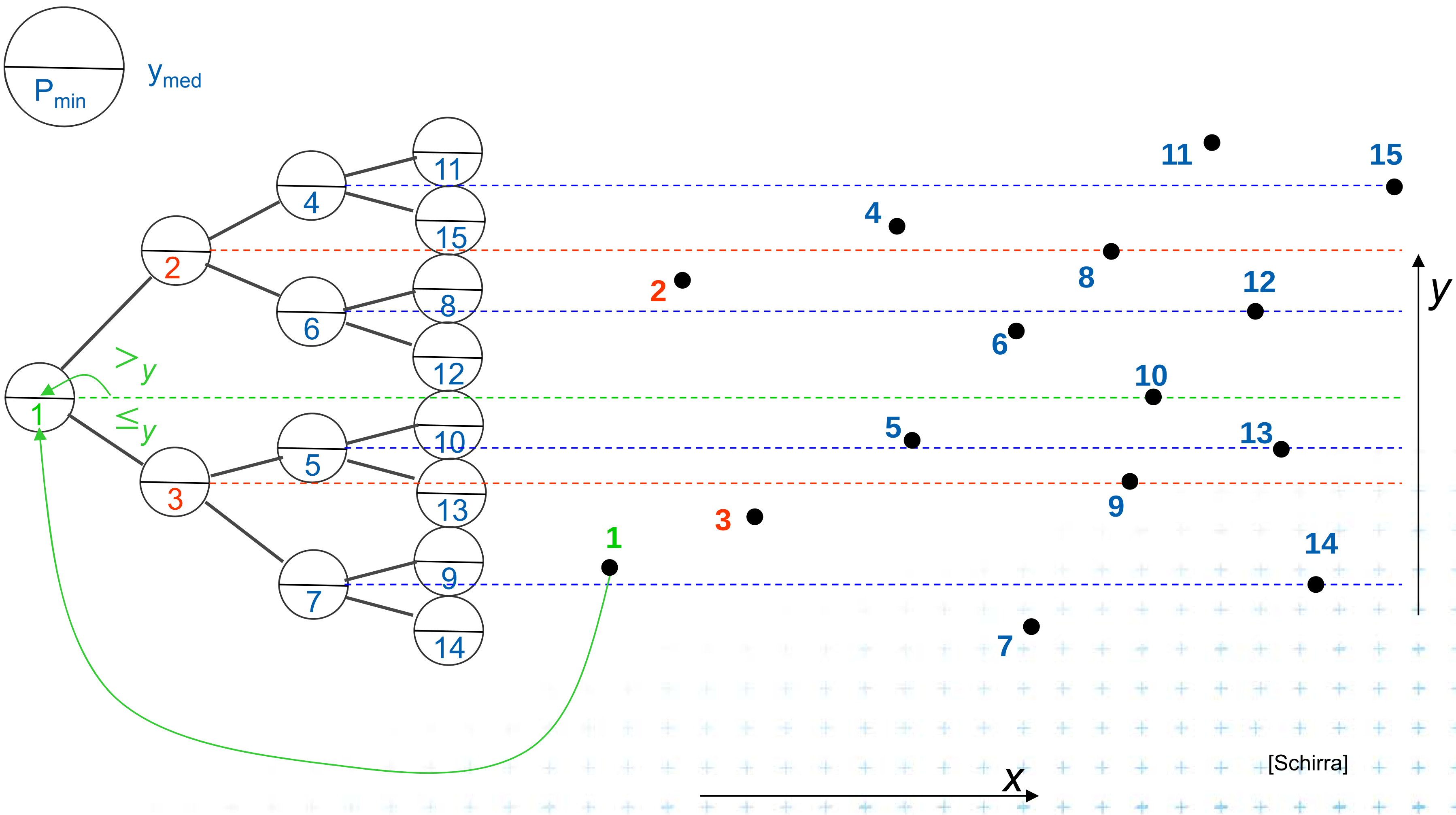
… p(v)

… y(v)

… lc(v)

… rc(v)

Priority search tree construction example



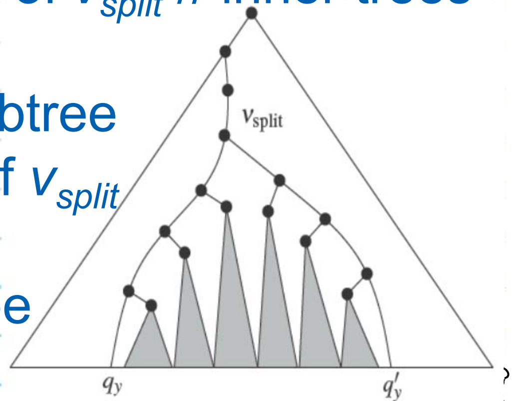
Query Priority Search Tree

QueryPrioritySearchTree(T, $(-\infty : q_x] \times [q_y \, ; q'_y\,]$ **)**
Input: A priority search tree and a range, unbounded to the left
Output: All points lying in the range

1. Search with q_y and q'_y in T // BST on y-coordinate – select y range
   Let v_{split} be the node where the two search paths split (split node)

2. for each node v on the search path of q_y or q'_y // points along the paths
3. if $p(v) \in (-\infty : q_x] \times [q_y \, ; q'_y\,]$ then report $p(v)$ // starting in tree root

4. for each node v on the path of q_y in the left subtree of v_{split} // inner trees
5. if the search path goes left at v
6. ReportInSubtree($rc(v)$, q_x) // report right subtree
7. for each node v on the path of q'_y in right subtree of v_{split}
8. if the search path goes right at v
9. ReportInSubtree($lc(v)$, q_x) // rep. left subtree



Reporting of subtrees between the paths

ReportInSubtree(v, q_x)

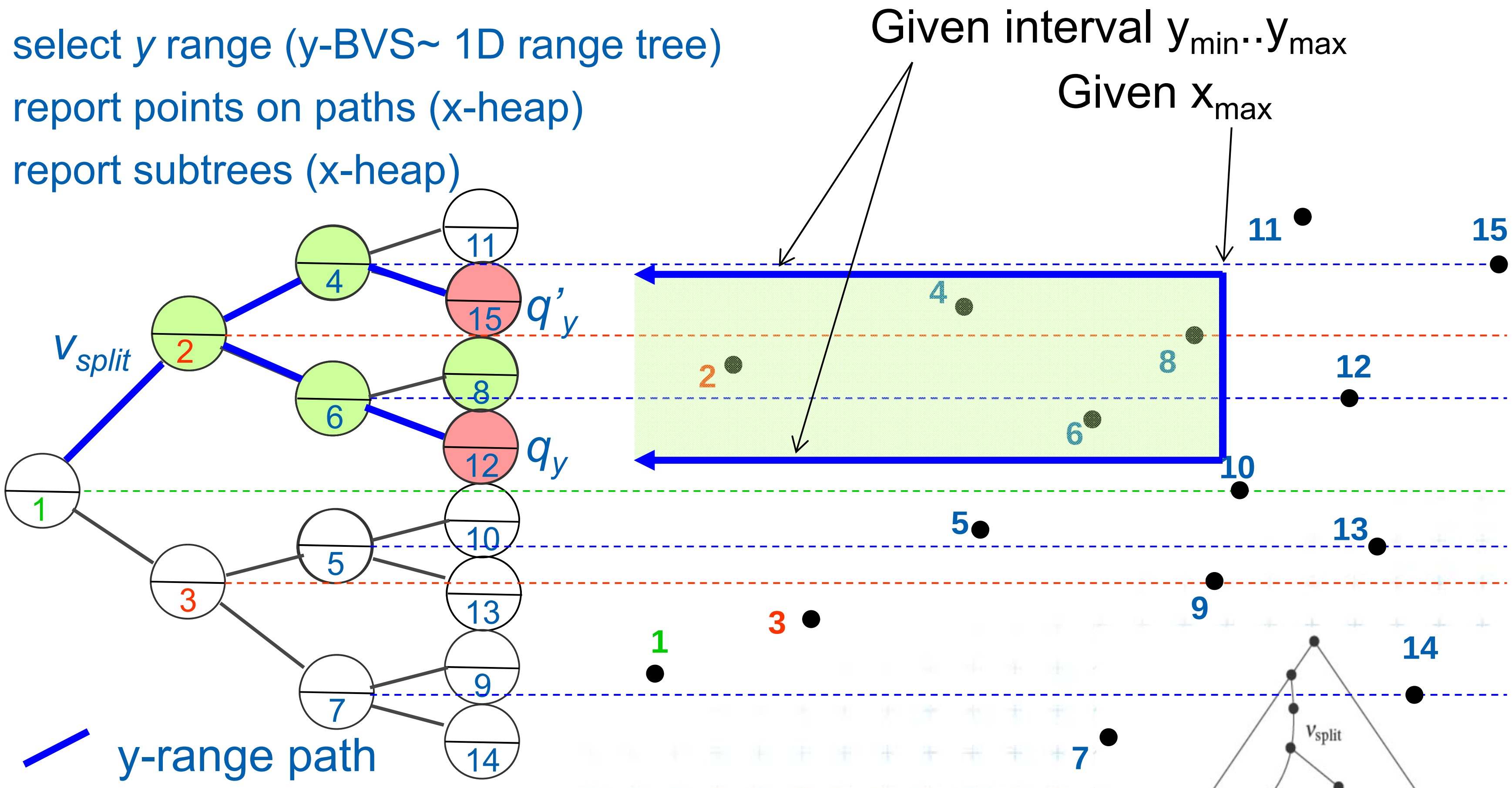
Input: The root v of a subtree of a priority search tree and a value q_x.

Output: All points in the subtree with x-coordinate at most q_x.

1. if v is not a leaf and $x(p(v)) \leq q_x$ // $x \in (-\infty : q_x]$
2. Report $p(v)$.
3. ReportInSubtree($lc(v)$, q_x)
4. ReportInSubtree($rc(v)$, q_x)

Priority search tree query

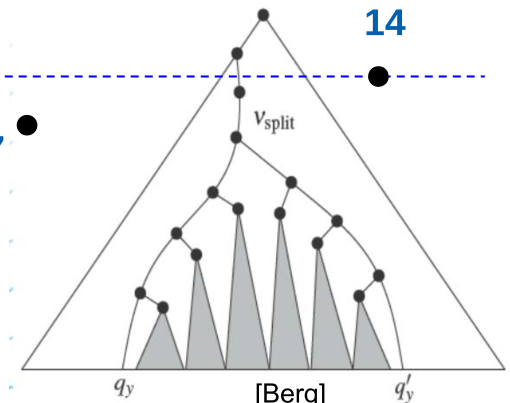
1. select *y* range (y-BVS~ 1D range tree)
2. report points on paths (x-heap)
3. report subtrees (x-heap)



y-range path

x ok – report this point

x too high – stop



[Berg]

Based on [Schirra]

Priority search tree complexity

For set of *n* points in the plane

- Build $O(n \log n)$
- Storage $O(n)$
- Query $O(k + \log n)$
  - points in query range $(-\infty : q_x] \times [q_y ; q'_y])$
  - k is number of reported points

- Use PST as associated data structure for interval trees for storage of M

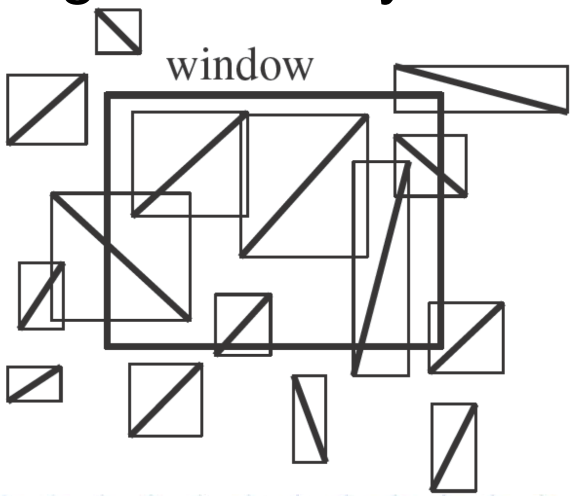
Windowing of arbitrary oriented line segments

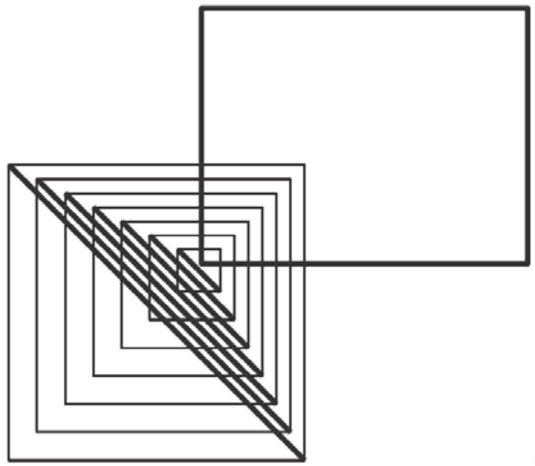
- Two cases of intersection

  a,b) Endpoint inside the query window => range tree

  c) Segment intersects side of query window => ???

- Intersection with BBOX?
  - Intersection with 4n sides
  - But segments may not intersect the window

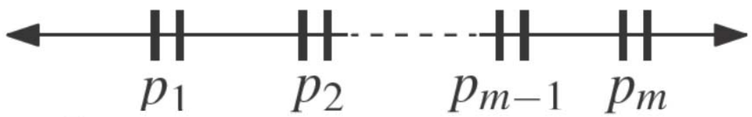




Segment tree

[Bentley, 1977]

- Exploits locus approach
  - Partition parameter space into regions of same answer
  - Localization of such region = knowing the answer
- For given set S of n intervals (segments) on real line
  - Finds m elementary intervals (induced by interval end-points)
  - Partitions 1D parameter space into these elementary intervals



$$(-\infty : p_1), [p_1 : p_1], (p_1 : p_2), [p_2 : p_2], \ldots, (p_{m-1} : p_m), [p_m : p_m], (p_m : +\infty)$$

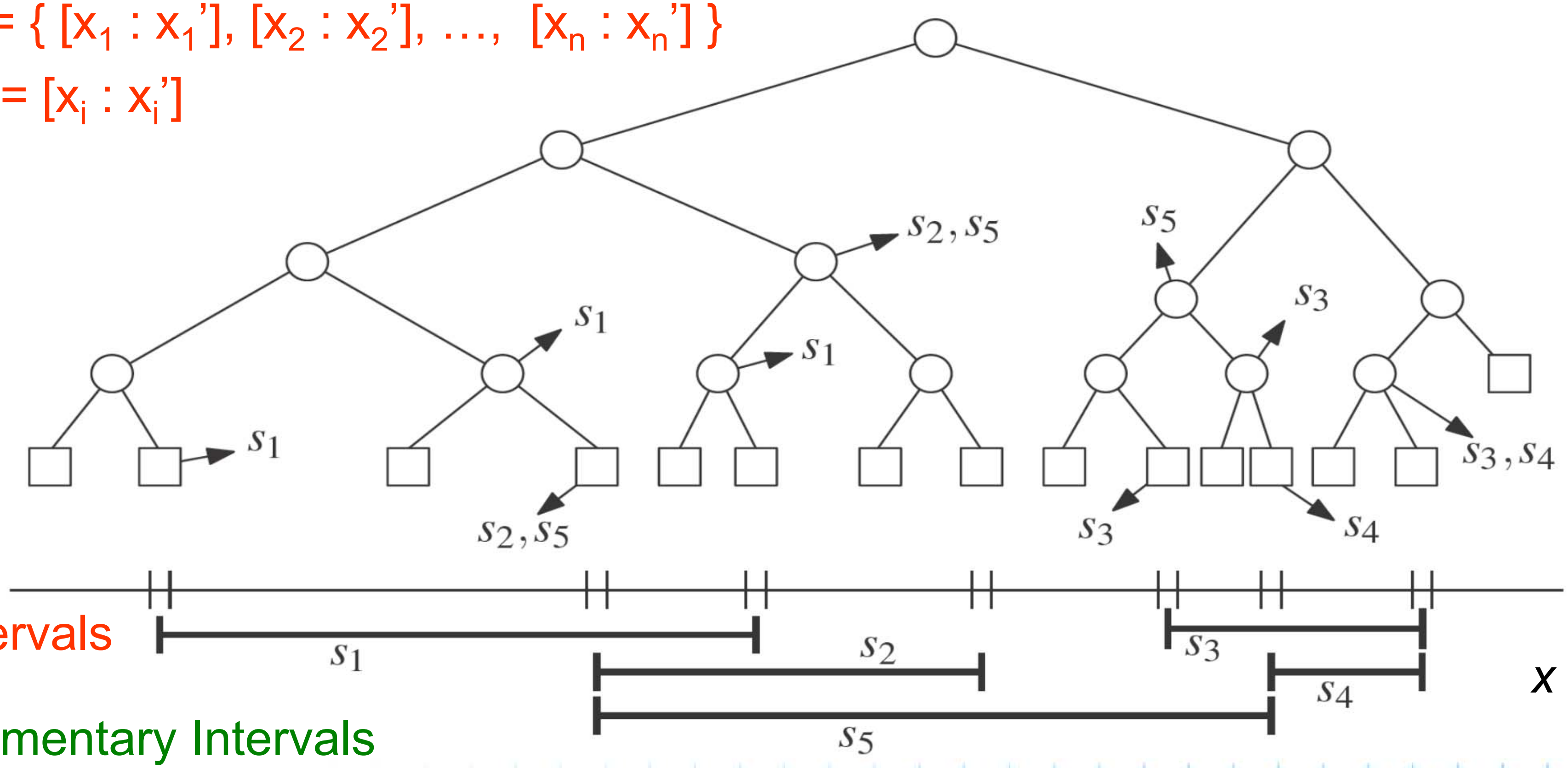
  - Stores intervals s_i with the elementary intervals
  - Reports the intervals s_i containing query point q_x.

Segment tree example

Intervals

$S = \{ [x_1 : x_1'], [x_2 : x_2'], \ldots, \; [x_n : x_n'] \}$

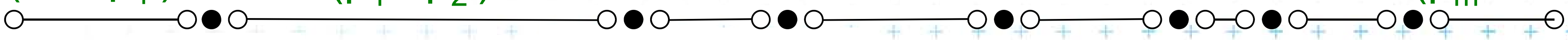
$s_i \; = [x_i : x_i']$



Intervals

Elementary Intervals

$(-\infty : p_1)$ $\quad$ $(p_1 : p_2)$ $\quad \ldots \quad$ $(p_m : +\infty)$



$[p_1 : p_1]$ ______ $[p_2 : p_2]$ $[p_2 : p_3]$ ______ …

DCGI

Segment tree definition

Segment tree

- Skeleton is a balanced binary tree T
- Leaves ~ elementary intervals Int(v)
- Internal nodes v
  ~ union of elementary intervals of its children
  - Store: 1. interval Int(v) = union of elementary intervals of its children
    2. canonical set $S(v)$ of intervals $[x : x'] \in S$ (segments s_i)
  - Holds $\text{Int}(v) \subseteq [x : x']$ and $\text{Int}(\text{parent}(v)) \not\subseteq [x : x']$ (node interval is not larger than a segment)
  - Intervals $[x : x']$ are stored as high as possible, such that $\text{Int}(v)$ is completely contained in the segment

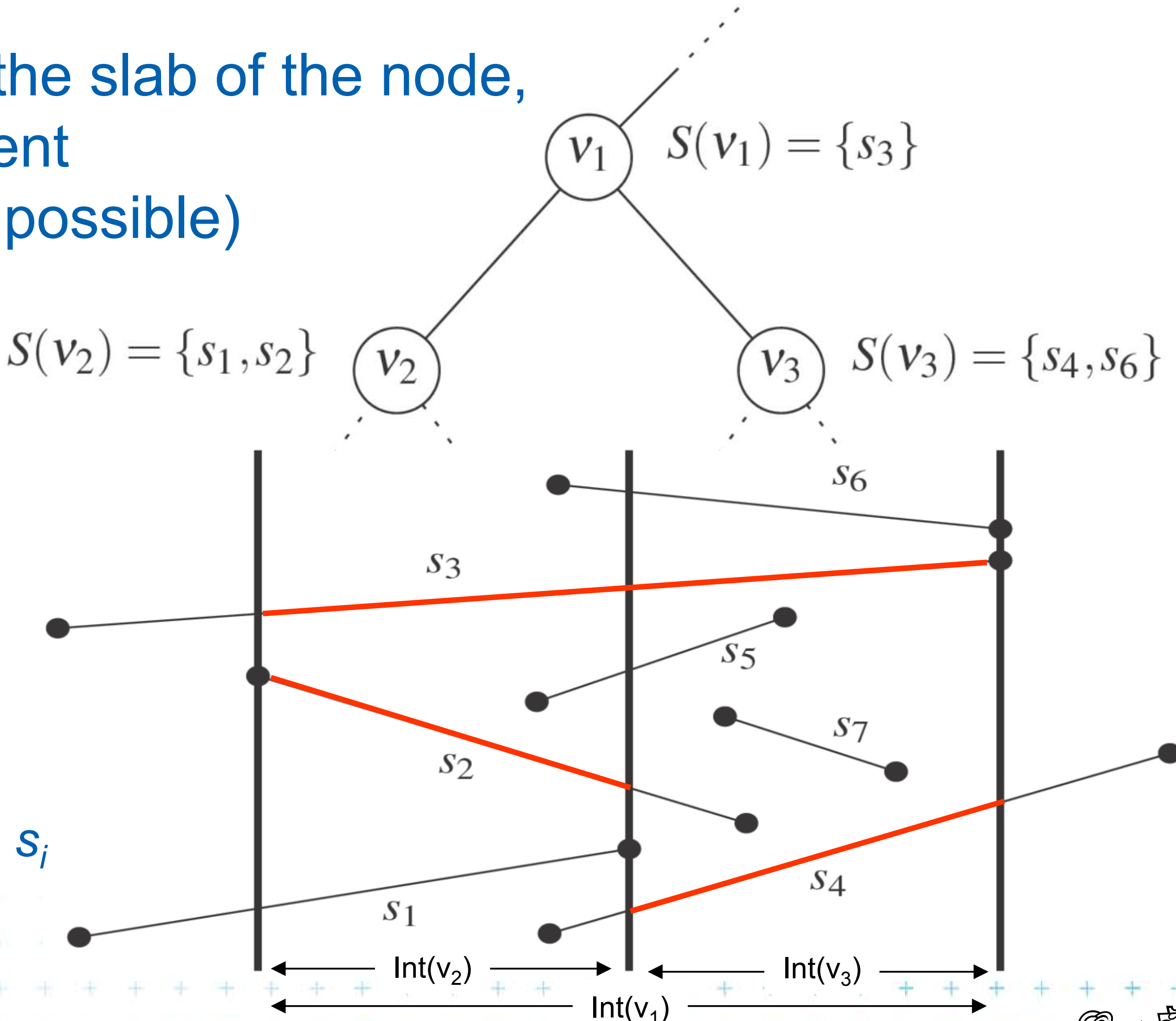
Segments span the slab

Segments span the slab of the node,
but not of its parent
(stored as up as possible)

$\mathrm{Int}(v_j) \subseteq s_i$

and

$\mathrm{Int}(\mathrm{parent}(v_j)] \nsubseteq s_i$



Query segment tree

QuerySegmentTree(v, q_x)
Input: The root of a (subtree of a) segment tree and a query point q_x
Output: All intervals in the tree containing q_x.

1. Report all the intervals s_i in $S(v)$.
2. **if** v is not a leaf
3. if $q_x \in$ Int($lc(v)$)
4. QuerySegmentTree($lc(v)$, q_x)
5. else
6. QuerySegmentTree($rc(v)$, q_x)

Query time O(log n + k), where k is the number of reported intervals
Height O(log n), O(1 + k_v) for node
Storage O(n log n)

Segment tree construction

ConstructSegmentTree(S)
Input: Set of intervals S - segments
Output: segment tree

1. Sort endpoints of segments in S -> get elemetary intervals …O(n log n)
2. Construct a binary search tree T on elementary intervals …O(n) (bottom up) and determine the interval Int(v) it represents
3. Compute the canonical subsets for the nodes (lists of their segments):
4. v = root(T)
5. for all segments $s_i = [x : x'] \in S$
6. InsertSegmentTree(v, $[x : x']$)

Segment tree construction – interval insertion

InsertSegmentTree(v, $[x : x']$)

Input: The root of a (subtree of a) segment tree and an interval.

Output: The interval will be stored in the subtree.

1. **if** Int(v) $\subseteq [\, x : x' \,]$ *// Int(v) contains $s_i = [\, x : x' \,]$*
2. store $[\, x : x' \,]$ at v
3. **else if** Int($lc(v)$) $\cap [\, x : x' \,] \neq \emptyset$
4. InsertSegmentTree($lc(v)$, $[x : x' \,]$)
5. **if** Int($rc(v)$) $\cap [\, x : x' \,] \neq \emptyset$
6. InsertSegmentTree($rc(v)$, $[x : x' \,]$)

One interval is stored at most twice in one level =>

Single interval insert O(log n)

Construction total O(n log n)

Segment tree complexity

A segment tree for set *S* of *n* intervals in the plane,

- Build $O(n \log n)$
- Storage $O(n \log n)$
- Query $O(k + \log n)$
  - Report all intervals that contain a query point
  - *k* is number of reported intervals

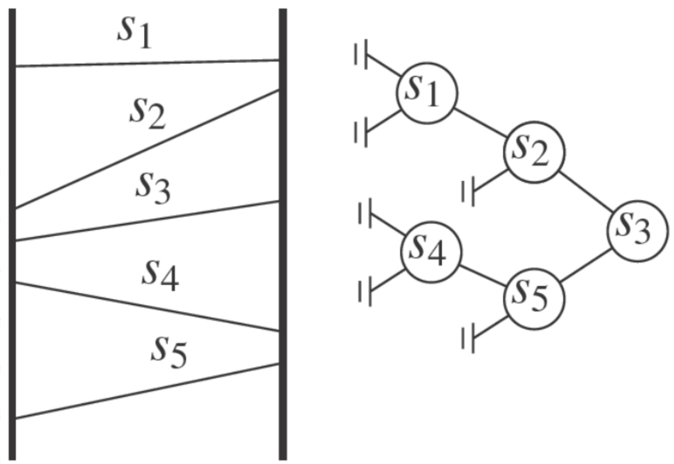
Segment tree versus Interval tree

- Segment tree
  - O(*n* log *n*) storage x O(*n*) of Interval tree
  - But returns exactly the intersected segments s_i, interval tree must search the lists ML and/or MR
- Good for
  1. extensions (allows different structuring of intervals)
  2. stabbing counting queries
     – store number of intersected intervals in nodes
     – O(n) storage and O(log *n*) query time = optimal
  3. higher dimensions – multilevel segment trees
     (Interval and priority search trees do not exist in ^dims)

Windowing of arbitrary oriented line segments

- Let S be a set of arbitrarily oriented line segments in the plane.
- Report the segments intersecting a vertical query segment $q := q_x \times [q_y : q'_y]$
- Segment tree T on x intervals of segments in S
  - node v of T corresponds to vertical slab $\text{Int}(v) \times (-\infty : \infty)$
  - segments span the slab of the node, but not of its parent
  - segments do not intersect

  => segments can be vertically ordered in the slab – BST



Segments between vertical segment endpoints

- Segments (in the slab) do not mutually intersect

  => segments can be vertically ordered and stored in BST

  – Each node v of the segment tree has an associated BST

  – BST $T(v)$ of node v stores the canonical subset $S(v)$ according to the vertical order

  – Intersected segments can be found by searching $T(v)$ in $O(k_v + \log n)$, k_v is the number of intersected segments

- Segment s is intersected by vert.query segment q iff

  – The lower endpoint of q is below s and

  – The upper endpoint of q is above s

Windowing complexity

Structure associated to node (BST) uses storage linear in the size of $S(v)$

- Build O(n log n)
- Storage O(n log n)
- Query O(k + $\log^2 n$)
  - Report all segments that contain a query point
  - k is number of reported segments

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