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# Image as Markov Random Field and Applications<sup>1</sup>

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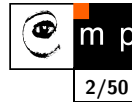
Talk Outline

Last update: December 2, 2010

- ◆ intro, applications
- ◆ MRF, labeling . . .
- ◆ how it can be computed at all?
- ◆ Applications in segmentation: GraphCut, GrabCut, demos

<sup>1</sup>Please note that the lecture will be accompanied by several sketches and derivations on the blackboard and few live-interactive demos in Matlab

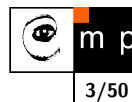
## About this very lecture



few notes before we start

- ◆ MRF is a complicated topic
- ◆ this lecture is introductory
- ◆ some simplifications in order not to lose the whole picture
- ◆ most important references provided
- ◆ many accessible explanations on the web (wikipedia . . . )

## Markov Random Field – MRF



From wikipedia<sup>2</sup>:

A Markov random field, Markov network or undirected graphical model is a graphical model in which a set of **random variables** have a **Markov property** described by an **undirected graph**.

More formal definition, which we follow, can be found in the first chapter<sup>3</sup> of the book [4].

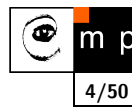
Let think about **images**:

- ◆ image intensities are the **random variables**
- ◆ the values depend only on their immediate spatial neighborhood which is the **Markov property**
- ◆ images are organized in a regular grid which can be seen as an **undirected graph**

<sup>2</sup>[http://en.wikipedia.org/wiki/Markov\\_random\\_field](http://en.wikipedia.org/wiki/Markov_random_field)

<sup>3</sup>freely available at: [http://www.nlpr.ia.ac.cn/users/szli/MRF\\_Book/MRF\\_Book.html](http://www.nlpr.ia.ac.cn/users/szli/MRF_Book/MRF_Book.html)

## Labeling for image analysis



Many image analysis and interpretation problems can be posed as **labeling problems**.

- ◆ Assign a **label** to image pixel (or to features in general).
- ◆ Image intensity can be considered as a label (think about palette images).

### Sites

$\mathcal{S}$  index a discrete set of  $m$  **sites**.

$$\mathcal{S} = \{1, \dots, m\}$$

Site could be:

- ◆ individual pixel
- ◆ image region
- ◆ corner point, line segment, surface patch . . .

## Labels and sites



A **label** is an event that may happen to a **site**.

Set of labels  $\mathcal{L}$ .

We will discuss **discrete labels**:

$$\mathcal{L} = \{\ell_1, \dots, \ell_M\}$$

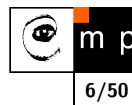
shortly

$$\mathcal{L} = \{1, \dots, M\}$$

### What can be a label?

- ◆ intensity value
- ◆ object label
- ◆ in edge detection binary flag  $\mathcal{L} = \{\text{edge}, \text{nonedge}\}$
- ◆ . . .

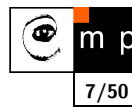
## Ordering of labels



Some labels can be ordered some not.

Ordered labels can be used to measure **distance** between labels.

# The Labeling Problem



Assigning a label from the label set  $\mathcal{L}$  to each of the sites  $\mathcal{S}$ .

## Example: edge detection in an image

Assign a label  $f_i$  from the set  $\mathcal{L} = \{\text{edge}, \text{nonedge}\}$  to site  $i \in \mathcal{S}$  where the elements in  $\mathcal{S}$  index the image pixels. The set

$$f = \{f_1, \dots, f_m\}$$

is called a **labeling**.

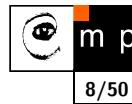
## Unique labels

When each site is assigned a unique label, labeling can be seen as **mapping** from  $\mathcal{S}$  to  $\mathcal{L}$ .

$$f : \mathcal{S} \longrightarrow \mathcal{L}$$

A labeling is also called a **coloring** in mathematical programming.

## How many possible labelings?



Assuming all  $m$  sites have the same label set  $\mathcal{L}$

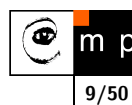
$$\mathbb{F} = \underbrace{\mathcal{L} \times \mathcal{L} \times \dots \times \mathcal{L}}_{m \text{ times}} = \mathcal{L}^m$$

Imagine the **image restoration** problem [3].

The  $m$  is the number of pixels in the image and  $\mathcal{L}$  equals to number of intensity levels.

Many, many possible labelings. Usually only few are good.

## Labeling problems in image analysis



- ◆ image restoration
- ◆ region segmentation
- ◆ edge detection
- ◆ object detection and recognition
- ◆ stereo
- ◆ ...

In images, site neighborhood matters.

A probability  $P(f_i)$  does not depend only on the site but also on the labeling around. Mathematically speaking we must consider conditional probability

$$P(f_i|\{f_{i'}\})$$

where  $\{f_{i'}\}$  denotes the set of other labels.

**no context:**

$$P(f) = \prod_{i \in \mathcal{S}} P(f_i)$$

**Markov Random Field - MRF**

$$P(f_i|f_{\mathcal{S}-\{i\}}) = P(f_i|f_{\mathcal{N}_i})$$

where  $f_{\mathcal{N}_i}$  stands for the labels at the sites **neighboring**  $i$ .

## MRF a Gibbs Random Fields

How to specify an MRF: in terms of conditional probabilities  $P(f_i|f_{\mathcal{N}_i})$  or joint probability  $P(f)$ ?

Hammersley–Clifford theorem about equivalence between MRF and Gibbs distribution.

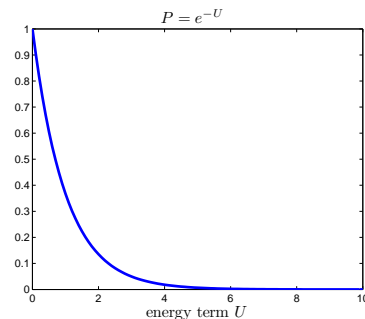
A set of random variables  $F$  is said to be a **Gibbs Random Fields** on  $\mathcal{S}$  with respect to  $\mathcal{N}$  iff its configurations obey a **Gibbs distribution**

$$P(f) = Z^{-1} \times e^{-\frac{1}{T}U(f)}$$

where

$$Z = \sum_{f \in \mathbb{F}} e^{-\frac{1}{T}U(f)}$$

is a normalizing constant.

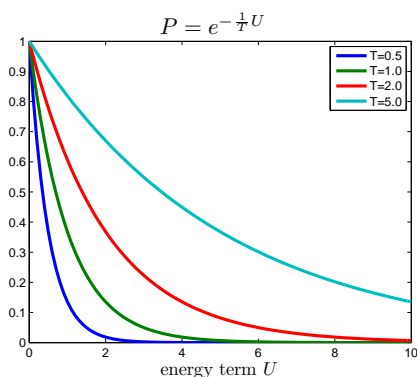


## Gibbs distribution

$$P(f) = Z^{-1} \times e^{-\frac{1}{T}U(f)}$$

◆  $T$  is **temperature**,  $T = 1$  unless stated otherwise

◆  $U(f)$  is the **energy function**



# Energy function

$$U(f) = \sum_{c \in \mathcal{C}} V_c(f)$$

is a sum of **clique potentials**  $V_c(f)$  over all possible cliques  $\mathcal{C}$ .

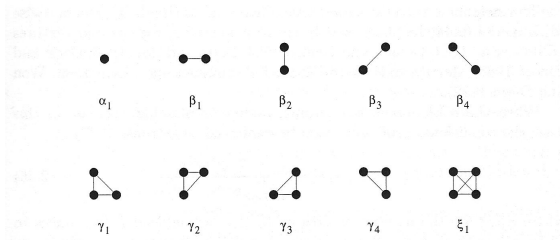


Figure 2.3: Clique types and associated potential parameters for the second-order neighborhood system. Sites are shown as dots and neighboring relationships as joining lines.

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<sup>4</sup>Illustration from the book [4]

## Simple cliques, Auto-models

Contextual constraints on two labels

$$U(f) = \sum_{i \in \mathcal{S}} V_1(f_i) + \sum_{i \in \mathcal{S}} \sum_{i' \in \mathcal{N}_i} V_2(f_i, f_{i'})$$

This can be interpreted as

$$U(f) = U_{data}(f) + U_{smooth}(f)$$

## Neighborhood and cliques on a regular lattices

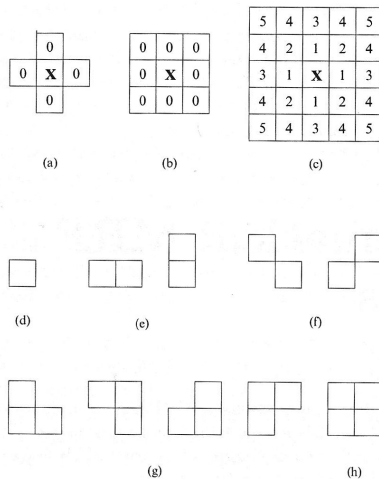
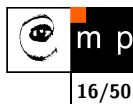


Figure 2.1: Neighborhood and cliques on a lattice of regular sites. 5

<sup>5</sup>Illustration from the book [4]

## Energy minimization



What is to be computed? Gibbs distribution:

$$P(f) = Z^{-1} \times e^{-\frac{1}{T}U(f)}$$

Remind that  $f$  is the desired labeling

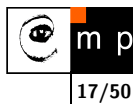
$$f : \mathcal{S} \rightarrow \mathcal{L}$$

In MAP formulation we seek the most probable labeling  $P(f)$ .

The best labeling minimizes energy  $U(f)$

It is a combinatorial problem. The next explanation follows mainly [2].

## Energy – data term and smoothness term



$$U(f) = U_{data}(f) + U_{smooth}(f)$$

The **data term** measures how well a label  $f_p$  fits the particular pixel (site)  $p$ . Globally,

$$U_{data}(f) = \sum_{p \in \mathcal{P}} D_p(f_p)$$

Example: in Image restoration  $D_p(f_p) = (f_p - I_p)^2$ , where  $I_p$  is the observed intensity and  $f_p$  is the label (assigned intensity).

### Smoothness term

Expresses the context. Setting a proper smoothness term is much more tricky.

- ◆ smooth but not everywhere (think about object boundary)
- ◆ **discontinuity preserving**

## Interacting pixels



We consider the energy

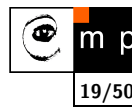
$$U(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{\{p,q\} \in \mathcal{N}} V_{p,q}(f_p, f_q)$$

where  $\mathcal{N}$  is the set of interacting pixels, typically adjacent pixels.  $D_p$  is assumed to be nonnegative.

Interaction of adjacent pixels may have long range impact!

$V_{p,q}$  is called **interaction penalty**.

## (reasonable) interaction penalties



labels  $\alpha, \beta, \gamma \in \mathcal{L}$

$$V(\alpha, \beta) = 0 \Leftrightarrow \alpha = \beta, \quad (1)$$

$$V(\alpha, \beta) = V(\beta, \alpha) \geq 0, \quad (2)$$

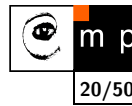
$$V(\alpha, \beta) \leq V(\alpha, \gamma) + V(\gamma, \beta) \quad (3)$$

Penalty is **metric** if all hold and **semimetric** if only (1,2) is satisfied.

### Examples of discontinuity preserving penalties

- ◆ truncated quadratic  $V(\alpha, \beta) = \min(K, (\alpha - \beta)^2)$
- ◆ truncated absolute distance  $V(\alpha, \beta) = \min(K, |\alpha - \beta|)$
- ◆ **Potts model**  $V(\alpha, \beta) = KT(\alpha \neq \beta)$ , where  $T() = 1$  if argument is true, otherwise 0.

## towards the best labeling (minimum energy)



Catch: finding global minimum is NP complete even for the simple Potts model.

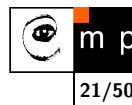
→ local minimum is sought.

### Problem

If the solution is poor

- ◆ poor choice of energy function
- ◆ local minimum is far from the global one

## Local minimum



a labeling  $f$  is a local minimum of the energy  $U$  if

$$U(f) \leq U(f')$$

for any  $f'$  near to  $f$ . In case of discrete labeling near means withing **single** move of  $f$ .

Many local minimization method use **standard** moves, where only one pixel (site) may change label at a time.

### Example: greedy optimization

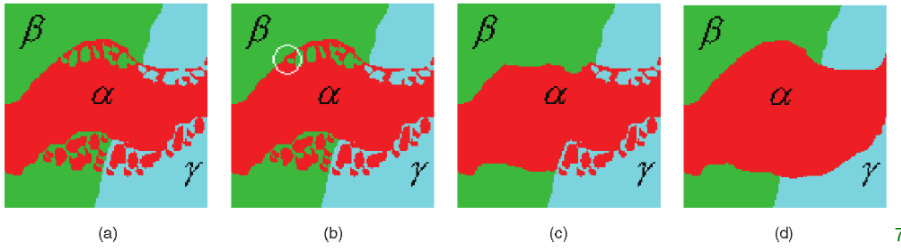
for each pixel, the label which gives the largest decrease of the energy is chosen.



# Fast approximate energy minimization via graph cuts

We only sketch the main ideas from the seminal work [2]<sup>6</sup>

Allow more than just one label change at a time



$\alpha$ - $\beta$  swap and  $\alpha$ -expansion.

<sup>6</sup>Freely available implementation. Many difficult problems in computer vision were solved by using this method and implementation.

<sup>7</sup>image from [2]

## Algorithm

1. Start with an arbitrary labeling  $f$
  2. Set success := 0
  3. For each pair of labels  $\{\alpha, \beta\} \subset \mathcal{L}$ 
    - 3.1. Find  $\hat{f} = \operatorname{argmin} E(f')$  among  $f'$  within one  $\alpha$ - $\beta$  swap of  $f$
    - 3.2. If  $E(\hat{f}) < E(f)$ , set  $f := \hat{f}$  and success := 1
  4. If success = 1 goto 2
  5. Return  $f$
- 
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## Algorithm

1. Start with an arbitrary labeling  $f$
  2. Set success := 0
  3. For each pair of labels  $\{\alpha, \beta\} \subset \mathcal{L}$  **iteration**
    - 3.1. Find  $\hat{f} = \operatorname{argmin} E(f')$  among  $f'$  within one  $\alpha$ - $\beta$  swap of  $f$
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1. Start with an arbitrary labeling  $f$  cycle
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3. For each pair of labels  $\{\alpha, \beta\} \subset \mathcal{L}$  iteration
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## $\alpha$ - $\beta$ swap

1. Start with an arbitrary labeling  $f$  cycle
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4. If  $\text{success} = 1$  goto 2
5. Return  $f$

- ◆ How many iterations in each cycle?
- ◆ A cycle is successful if a strictly better labeling is found in any iteration.
- ◆ Cycling stops after first unsuccessful cycle.

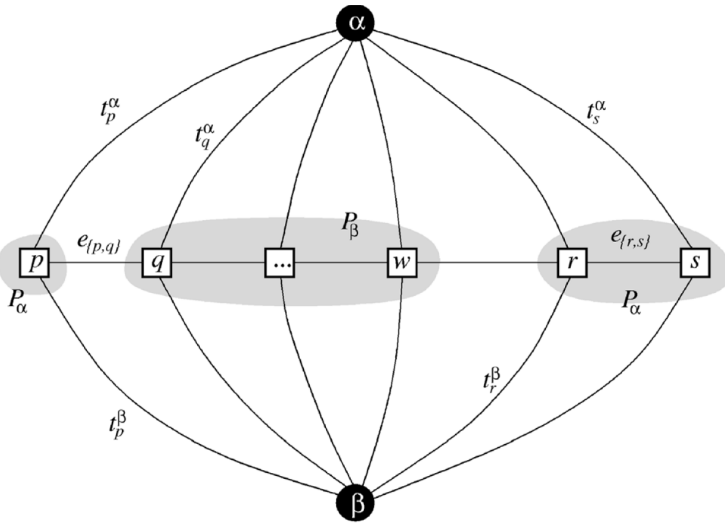
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4. If  $\text{success} = 1$  goto 2
5. Return  $f$

The best  $\alpha$ - $\beta$  swap is found by composing a graph and finding min-cut.

- ◆ min-cut is equivalent to max-flow (Ford-Fulkerson theorem)
- ◆ Max-flow between terminals is a standard problem in Combinatorial Optimization.
- ◆ Algorithms with low-order polynomial complexities exist.

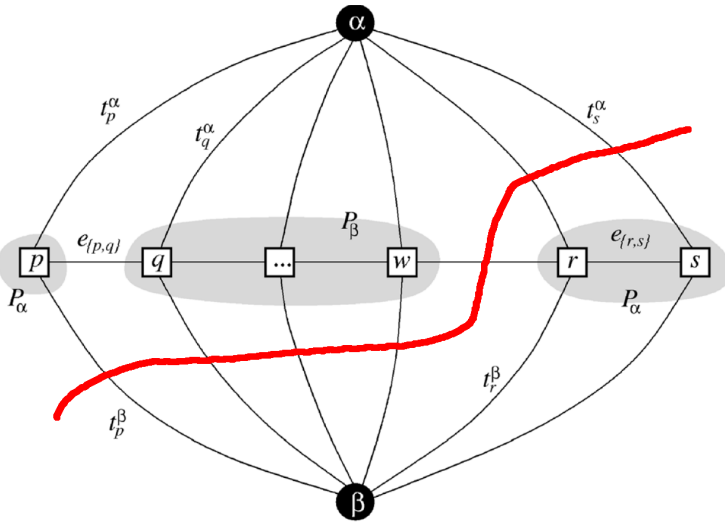
# Finding optimal $\alpha$ - $\beta$ swap



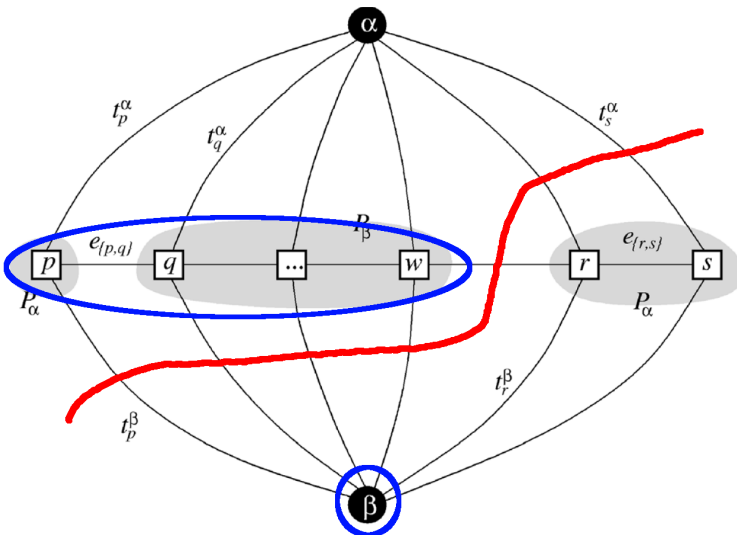
8

<sup>8</sup>illustration from [2]

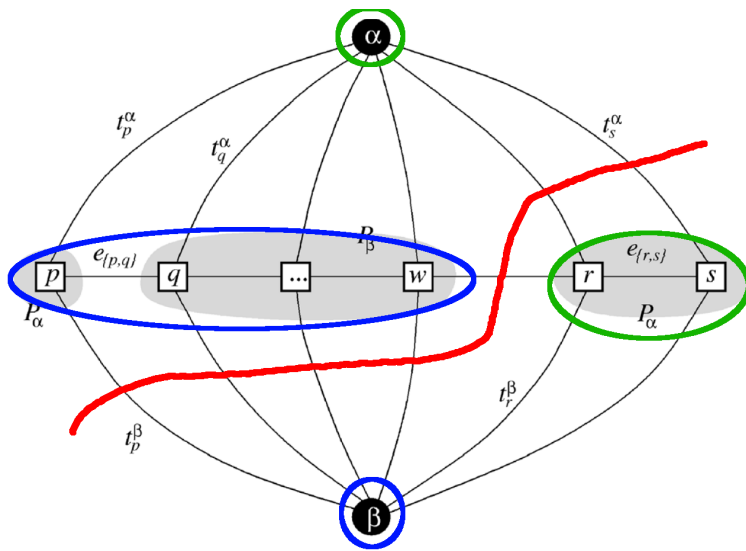
# Finding optimal $\alpha$ - $\beta$ swap, min-cut



# Finding optimal $\alpha$ - $\beta$ swap, re-labeling



## Finding optimal $\alpha$ - $\beta$ swap, re-labeling



## MRF in image segmentation

Zkouška

Pátek 4.6.

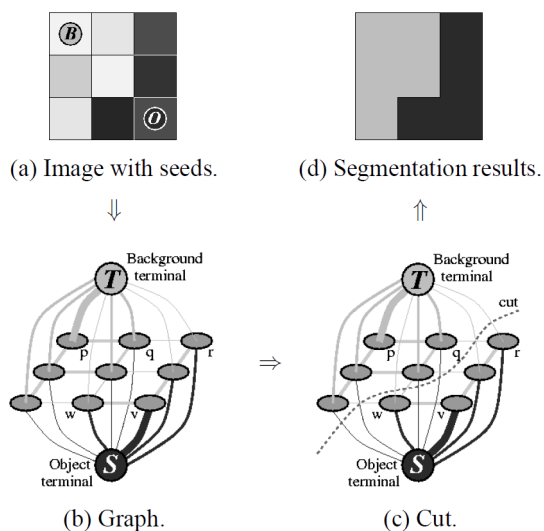
# Segmentation with seeds – Interactive GraphCuts

Idea: denote few pixels that trully belongs to object or background and than refine (grow) by using soft constraints.



- ◆ data term
- ◆ boundary penalties and/or pixel interactions
- ◆ how to find the optimal boundary between object and background

## Segmentation with seeds – graph



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<sup>9</sup>illustration from [1]

## Edges, cuts, segmentation

Explained on the blackboard.

See [1] for details.

## What data term?

$$U_{data}(obj, bck) = \sum_{p \in obj} D(obj) + \sum_{p \in bck} D(bck)$$

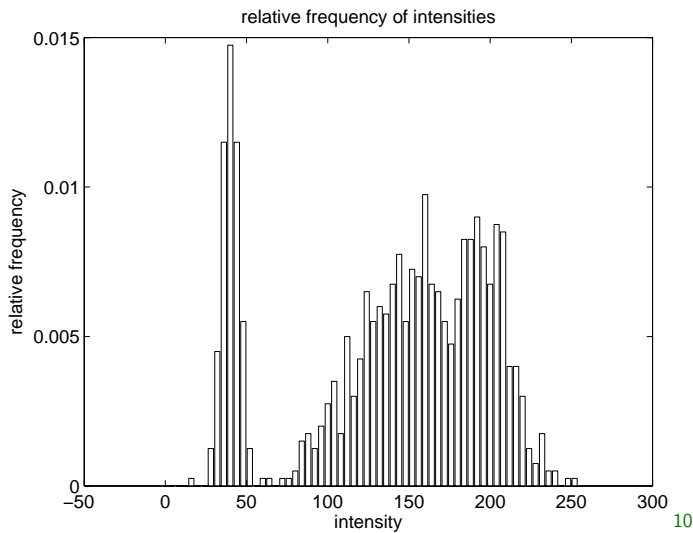
The better match between pixel and “obj” or “bck” model the lower energy (penalty).

### How to model?

for simplicity, you may think about the opposite case, the better match the higher value

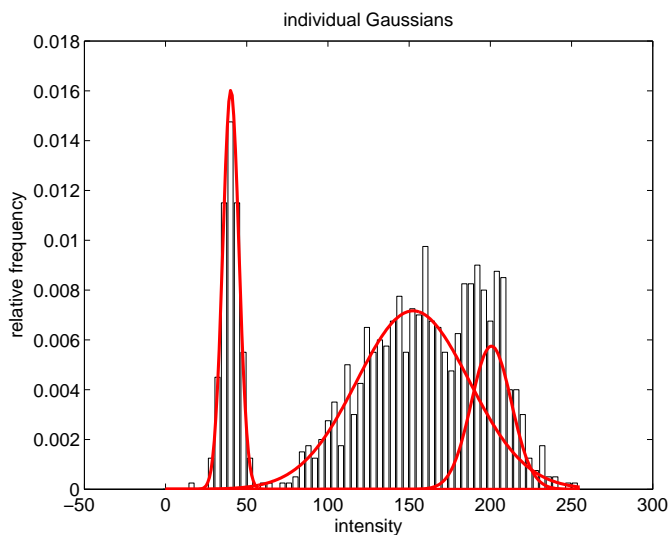
- ◆ background, foreground (object) pixels
- ◆ intensity or color distributions
- ◆ histograms
- ◆ parametric models: GMM – Gaussian Mixture Model

### Image intensities - 1D GMM

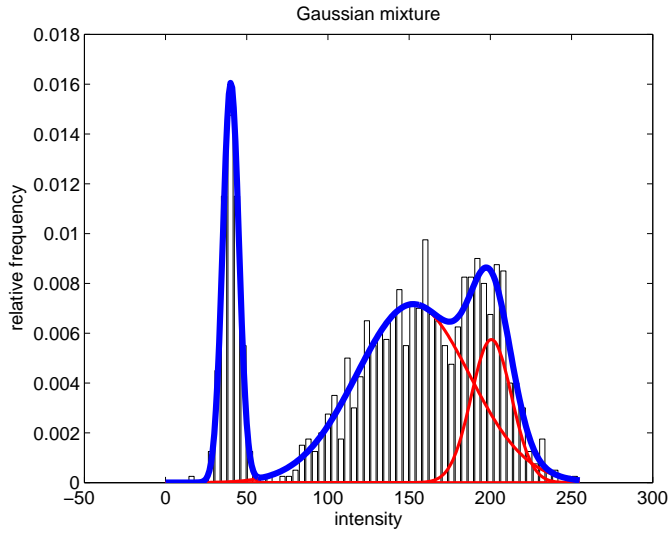


<sup>10</sup>Demo codes from [6]

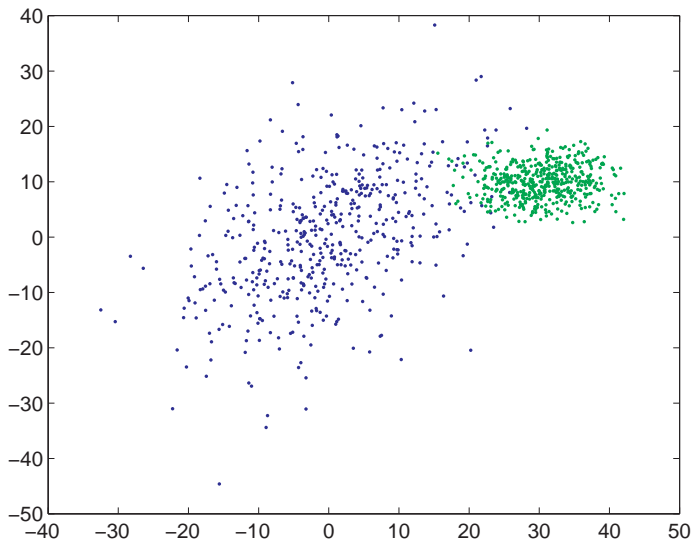
### Image intensities - 1D GMM



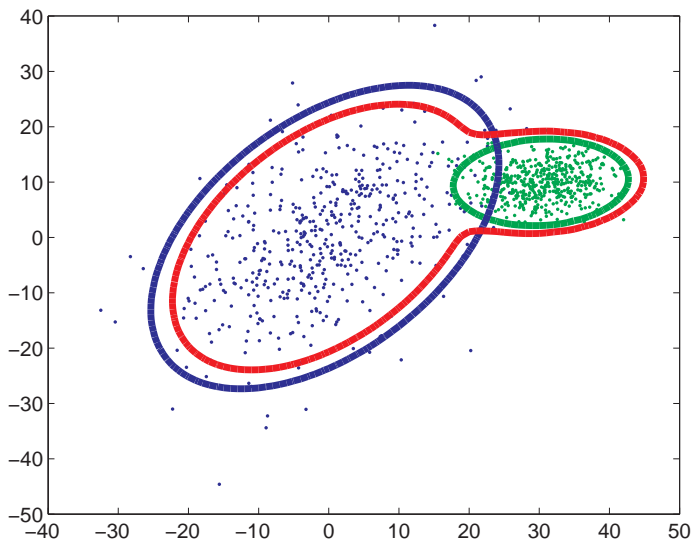
# Image intensities - 1D GMM

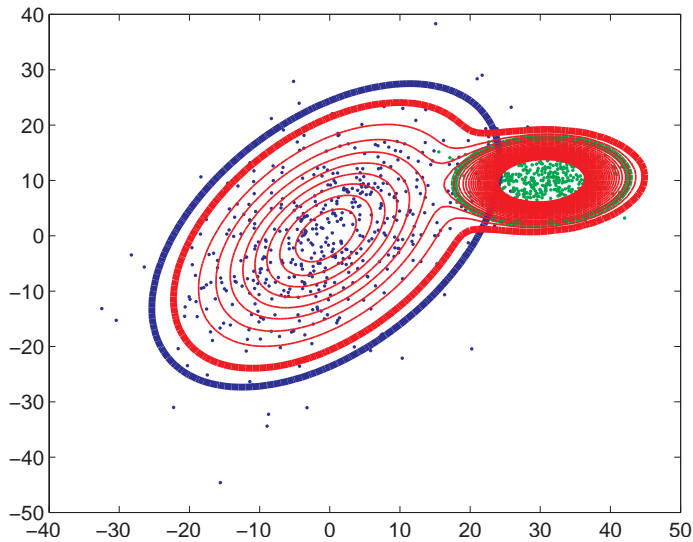


# 2D GMM



# 2D GMM





### Data term by GMM – math summary

$$p(\mathbf{x}|f_i) = \sum_{k=1}^K w_k^{f_i} \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma_k^{f_i}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu_k^{f_i})^T \Sigma_k^{f_i^{-1}} (\mathbf{x} - \mu_k^{f_i})\right),$$

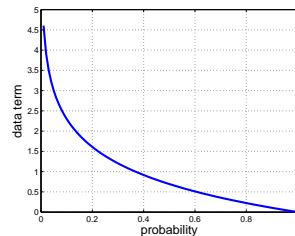
where  $d$  is the dimension.

- ◆  $K$  number of Gaussians. User defined.
- ◆ for each label  $\mathcal{L} = \{\text{obj}, \text{bck}\}$  different  $w_k, \mu_k, \Sigma_k$  estimated from the data (seeds)
- ◆  $\mathbf{x}$  pixel value, can be intensity, color vector, . . .

### Data term

$$D(\text{obj}) = -\ln p(\mathbf{x}|\text{obj})$$

$$D(\text{bck}) = -\ln p(\mathbf{x}|\text{bck})$$



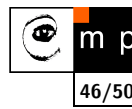
### Results data term only

See the live demo<sup>11</sup>

<sup>11</sup>Demo codes courtesy of V. Franc and A. Shekhovtsov



## What boundary (interaction) term?



The Potts model:  $V(p, q) = \lambda T(p \neq q)$ , where  $T() = 1$  if argument is true, otherwise 0.

Effect of  $\lambda$ , see the live demo.

### Results for data and interaction terms

## GrabCut – going beyond the seeds



The main idea: Iterate the graphcut and refine the data terms in each iteration. Stop if the energy (penalty) does not decrease. [5]

### The practical motivation

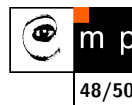
Further reduce the user interaction.



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<sup>12</sup>images from [5]

## GrabCut – algorithm



**Init:** Specify background pixels  $T_B$ .  $T_F = 0$ ;  $T_U = \bar{T}_B$

### Iterative minimization:

1. assign GMM components to pixels in  $T_U$
2. learn GMM parameters from data
3. segment by using min-cut (graphcut algorithm)
4. repeat from step 1 until convergence

Optional: edit some pixels and repeat the min-cut.

- [1] Yuri Boykov and Marie-Pierre Jolly. Interactive graph cuts for optimal boundary & region segmentation of objects in N-D images. In *Proceedings of International Conference on Computer Vision (ICCV)*, 2001.
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End



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