System for autonomous betting with optimal wealth allocation

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Outline

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Optimal Strategy

Coin Toss

- Favourable game
- $p = 0.4$, we get paid 3 times the bet amount. ($ev = 1.2$).
- Assume a "long run" of **approximately similar games**

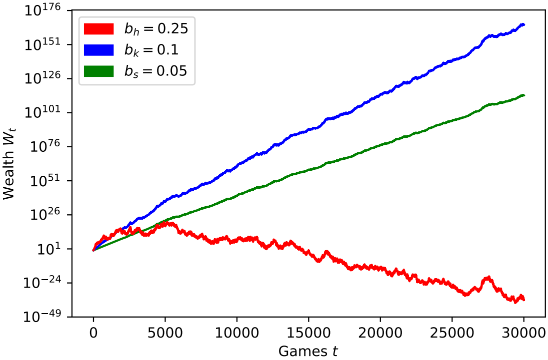
How much of our capital is this risky opportunity worth?

$$b_h = 0.25 \qquad b_k = 0.1 \qquad b_s = 0.05 \tag{1}$$

Optimal Strategy

Kelly Criterion

$$b^\star = \frac{edge}{odds} = \frac{0.4\cdot(3-1)-0.6}{(3-1)} = 0.1$$



- Kelly Criterion is the **upper bound** of value for each opportunity.
- Betting more $\rightarrow$ bankruptcy.
- Betting less $\rightarrow$ sub-optimal growth.

Optimal Strategy

General Case

- K probabilistic outcomes $p_1, p_2, ..., p_K$.
- n opportunities, (assets), $n-1$ risky, 1 risk-less. $\boldsymbol{a_1}, \boldsymbol{a_2}, ..., \boldsymbol{c}$.

$$\boldsymbol{p} = \begin{bmatrix} p_1 & p_2 & \dots & p_K \end{bmatrix} \qquad \boldsymbol{a_i} = \begin{bmatrix} r_{1,i} \\ r_{2,i} \\ \dots \\ r_{K,i} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} \tag{2}$$

Cash asset can have a different payoff if money can be **risk-free invested** elsewhere. (e.g. bank account interest rate)

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \dots & \boldsymbol{a_{n-1}} & \boldsymbol{c} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_c \end{bmatrix} \tag{3}$$

Example

Assume horse race with 16 running horses. Bet type quinella denoted $QNL(i,j)$ pays off if pair of horses (i,j) win the race. Order does not matter. There are hence 120 different pairs, 121 different assets including cash asset and 120 probabilities in the vector $\boldsymbol{p}$. $o_{i,j}$ denotes posted odds for given $QNL(i,j)$.

$$\boldsymbol{R} = \begin{bmatrix} o_{1,1} & 0 & 0 & ... & 1 \\ 0 & o_{1,2} & 0 & ... & 1 \\ 0 & 0 & o_{1,3} & ... & 1 \\ ... & ... & ... & ... & 1 \end{bmatrix} \tag{4}$$

This is a bet on an exclusive outcome, hence R matrix is almost completely made up of zeros and odds diagonally.

$$\boldsymbol{p} = \left[p_{1,1}, p_{1,2}, ..., p_{15,16}\right] \tag{5}$$

$$\boldsymbol{b} = \left[b_{1,1}, b_{1,2}, ..., b_c\right] \tag{6}$$

Optimal Strategy

General Case

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[\log(\boldsymbol{R} \cdot \boldsymbol{b})] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0,\ b_i \geq 0 \end{aligned}$$

The problem is similar to the **fractional Knapsack** problem formulation.

- Instead of diamonds and smaragds, risky opportunities are "put in the bag".
- Probabilistic outcomes with $\log(\mathit{payoff})$

Optimal Strategy

Economist's View

MPT states that portfolio $\boldsymbol{b_1}$ is superior to $\boldsymbol{b_2}$ if the expected gain $\mathbb{E}[\boldsymbol{b}]$ is at least as great.

$$\mathbb{E}[\boldsymbol{b_1}] \geq \mathbb{E}[\boldsymbol{b_2}] \tag{7}$$

and the risk, here general risk measure denoted r is no greater.

$$r(\boldsymbol{b_1}) \leq r(\boldsymbol{b_2}) \tag{8}$$

Simpler risk measures include the following:

$$Var[\boldsymbol{b}] \tag{9}$$

$$\sigma(\boldsymbol{b}) = \sqrt{Var[\boldsymbol{b}]} \tag{10}$$

$$CV(\boldsymbol{b}) = \frac{\sigma(\boldsymbol{b})}{\mathbb{E}[\boldsymbol{b}]} \tag{11}$$

Optimal Strategy

Modern Portfolio Theory

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \boldsymbol{\mu}^T \boldsymbol{b} - \gamma \boldsymbol{b}^T \Sigma \boldsymbol{b} \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0,\ b_i \geq 0 \end{aligned}$$

where $\boldsymbol{b}$ is fraction vector, γ is risk aversion parameter and $\boldsymbol{\mu}$ is the expected values vector of offered opportunities. In layman terms we maximize the following:

$$return - \gamma \cdot risk \tag{12}$$

In the most general set up risk is defined as variance Σ.

Optimal Strategy

Review

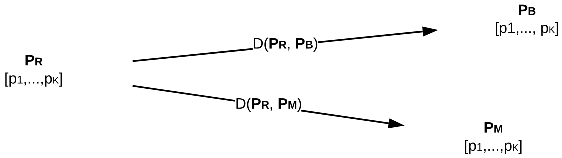
- Kelly is the **growth optimal** wealth allocation strategy if the following assumptions are met.
  1. True probability is known to the player.
  2. Approximately similar choice is to be made infinitely many times.
- There is closed form solution for exclusive games.
- No closed form for a general case.
- Solvable by SCS(Splitting Conic Solver)...
- Economists view the problem differently(Max return Min risk)

Uncertainty

- We do not know the true probability. $\rightarrow$ Neither does bookmaker.

Game Scenario

$$(P_R, P_B, P_M) \tag{13}$$



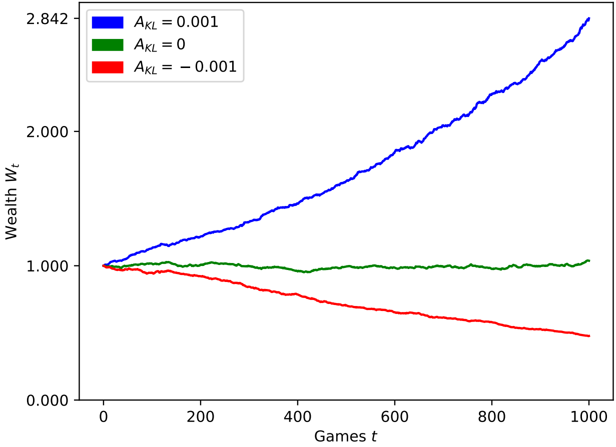
- P_R Real probability.
- P_B Bookie's estimate.
- P_M Model's estimate of the real probability.
- $D(P, Q)$ Kullback–Leibler divergence.

$$D_{KL}(P||Q) = \sum_{i=0}^{n} p_i \log \frac{p_i}{q_i} \tag{14}$$

Uncertainty

Information is Money (Kelly / Shannon)

- $A_{KL} = D(P_R||P_B) - D(P_R||P_M) \qquad CGR = A_{KL}$



Uncertainty

Drawdown

- Half-Kelly approach uses fixed fraction. (Too static).
- Dynamic approach is to add a drawdown constraint.

$$P(W^{MIN} < 0.7) \leq 0.1 \tag{15}$$

Probability of our wealth falling below 0.7 is $p \leq 0.1$, in general:

$$P(W^{MIN} < \alpha) \leq \beta \tag{16}$$

The drawdown constraint is approximately satisfied if the following is satisfied, Busseti et al., 2016.

$$\mathbb{E}[(\boldsymbol{R} \cdot \boldsymbol{b})^{-\lambda}] \leq 1 \tag{17}$$

Where

$$\lambda = \frac{\log(\beta)}{\log(\alpha)} \tag{18}$$

Uncertainty

Risk Constrained Kelly

$$\begin{aligned}
\underset{\boldsymbol{b}}{\text{maximize}} \quad & \sum_{i=1}^{K} p_i \cdot log(\boldsymbol{R_i} \cdot \boldsymbol{b}) \\
\text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\
& b_i \geq 0 \\
& \log(\sum_{i=1}^{K} \exp[\log(p_i) - \lambda \log(\boldsymbol{R_i} \cdot \boldsymbol{b})]) \leq 0
\end{aligned}$$

where $\lambda = \frac{\log(\beta)}{\log(\alpha)}$ for some $\alpha, \beta \in (0, 1)$

Result

Portfolio satisfies $P(W^{MIN} < \alpha) \leq \beta$ and is as "growth optimal" as possible.

Uncertainty

Distributionally robust optimization

Paradigm for decision making under uncertainty where:

- The uncertain problem data are governed by a probability distribution that is itself subject to uncertainty.
- The distribution is then assumed to belong to an ambiguity set comprising all distributions that are compatible with the decision maker's prior information.

Multiple different kinds of distribution sets for Kelly gambling.

- Ellipsoidal distribution set
- Polyhedral
- Divergence based(KL div)
- Wasserstein distance

Idea: Look at the worst case growth rate possible given the uncertainty.

Combinatorial Explosion

- 30 games $\rightarrow K = 3^{30}$ outcomes. $\boldsymbol{R} \rightarrow$ Combinatorial explosion
- **Logarithms** in objective function and constraints $\rightarrow$ numerically difficult.

Quadratic Kelly

1. Taylor expansion of both the objective f, drawdown constraint.
2. Take first two terms.

$\boldsymbol{\rho}$ is the matrix of excess return. $\boldsymbol{\rho} = \boldsymbol{R} - 1$

$$
\begin{aligned}
\underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[\boldsymbol{\rho} \cdot \boldsymbol{b}] - \frac{1}{2}\,\mathbb{E}[(\boldsymbol{\rho} \cdot \boldsymbol{b})^2] \\
\text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \qquad\qquad b_i \geq 0 \\
& \lambda(\lambda + 1)\frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} \leq \lambda(\boldsymbol{\rho} \cdot \boldsymbol{b}) \quad \lambda = \frac{\log(\beta)}{\log(\alpha)}
\end{aligned}
$$

Combinatorial Explosion

Outcomes ≤ 4

- Idea: Choose only a single asset per game, ignore all the others.
- Idea 2: Choose according to sharpe ratio. *reward* / *risk*.

Max Sharpe Ratio

$$
\begin{aligned}
\underset{\boldsymbol{b}}{\text{maximize}} \quad & \frac{\boldsymbol{\mu b}}{\sqrt{\boldsymbol{b}^T \boldsymbol{\Sigma b}}} \\
\text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\
& b_i \geq 0
\end{aligned}
$$

where $\boldsymbol{\Sigma}$ is a covariance matrix. $\boldsymbol{\mu}$ is a row vector of excess returns.

Open questions

What about machine learning?

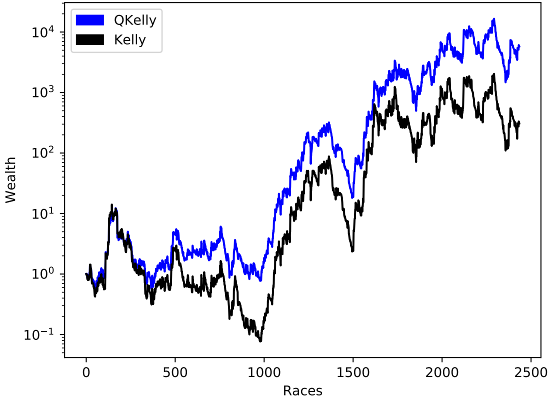
- Hyperparameter optimization
  - Simulations/validations are costly
  - Smarter parameter space exploration
  - Bayesian optimization methods
  - SMBO based on Gaussian Process
- Reinforcement learning?
  - Existing literature over simplifies the problem
  - Rarely good results
  - Often highest expectation bets strategy only etc.
- Bayesian policies
  - Games are often not approximately similar
  - A lot can be learned about the distribution of dividends.

Experiments

Horse Racing

- Korea $\rightarrow$ Seoul racetrack dataset

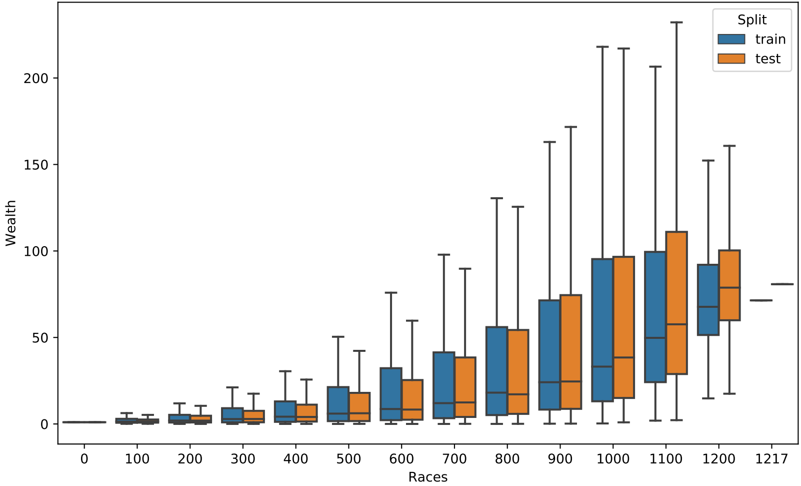
K	$odds$	$margin$	$A_{KL}/log(K)$
$\in [6, 16]$	$\in [1.0, 931.3]$	≈ 0.2	0.0057



Experiments

Horse Racing

Kelly more reasonable when constrained $\rightarrow \boldsymbol{P(W^{MIN} < 0.4) \leq 0.1}$.



Experiments

Basketball

K	$odds$	$margin$	$A_{KL}/log(K)$
2	$\in [1.01, 41.]$	≈ 0.04	-0.0146

- KL disadvantage $\rightarrow$ Kelly is not the optimal strategy.
- We assume 10-game rounds. $K_{round} = 2^{10}$
- Train and Test datasets always randomly shuffled, some randomly removed.

The risk constraint and fractional parameters are selected according to.

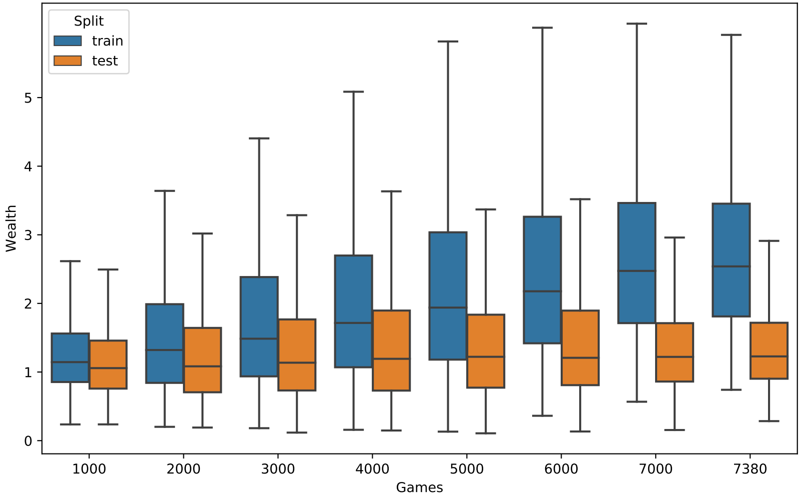
$$\begin{aligned} \text{maximize} \quad & median(W_F) \\ \text{subject to} \quad & Q_5 > 0.95 \end{aligned}$$

- Maximal median final wealth W_F
- The 5th percentile, value below which 5% of all the wealth positions may be found is greater than 0.95.

Experiments

Basketball

Fractional MaxSharpe

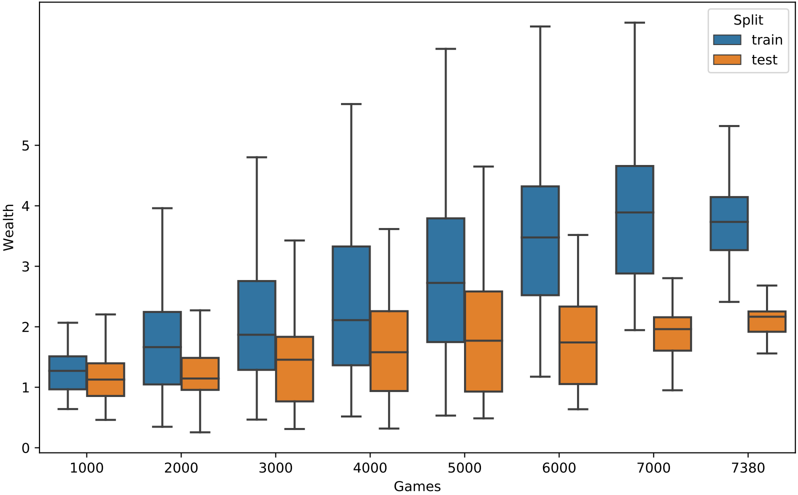


1000 MSharpe trajectories across the testing and training datasets

Experiments

Basketball

Quadratic Kelly, risk constrained.



Experiments

Football

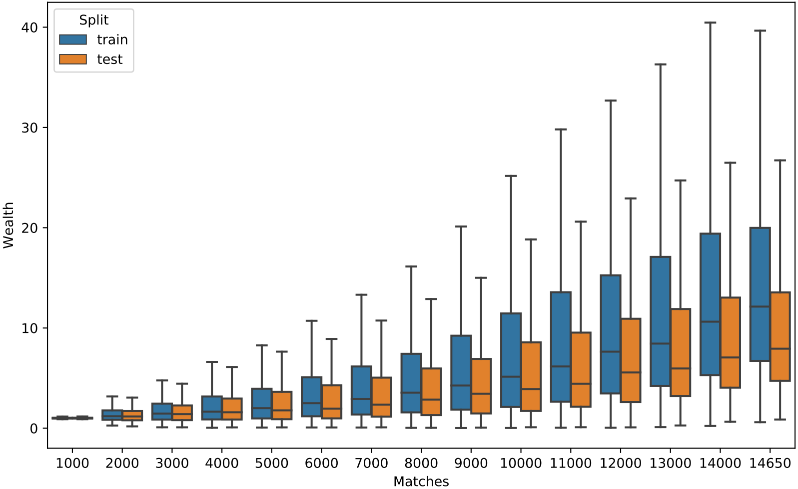
K	$odds$	$margin$	$A_{KL_o}/log(K)$	$A_{KL_c}/log(K)$
3	$\in [1.03, 66]$	0.03	-0.012	-0.016

- KL disadvantage $\rightarrow$ Kelly not optimal.
- $A_{KL_o} > A_{KL_c}$ $\rightarrow$ Opening odds more advantageous.
- We assume 10-game rounds. $K = 3^{10}$ $\rightarrow$ Select 2 $\rightarrow$ $K = 2^{10}$
- Train and Test datasets always randomly shuffled.
- Some games always randomly removed.
- The same criterion for parameters as basketball.

Experiments

Football Reinvestment

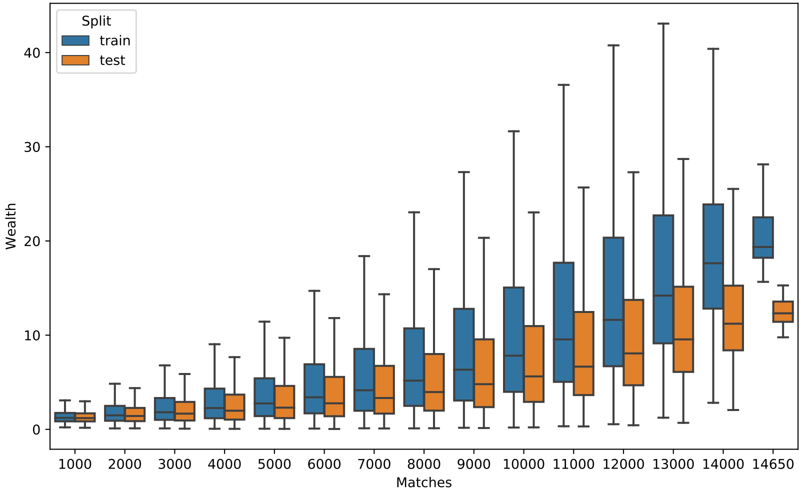
Fractional MaxSharpe



Experiments

Football Reinvestment

Quadratic Kelly risk constrained



Review

Findings

1. Kelly growth optimal under specific assumptions.
2. KL advantage directly connected to exponential growth rate.
3. Linear growth no such connection
4. Kelly numerically difficult
5. MPT is quadratic approximation of Kelly (scales badly for more difficult problems)
6. Under specific conditions it pays to bet negative EV
7. ...

Open questions

1. General advantage function?
2. How to apply ML?