Computing Correlated Equilibrium and Succinct Representation of Games

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Correlated Equilibrium

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Correlated Equilibrium – a probability distribution over pure strategy profiles $p = \Delta(\mathcal{S})$ that recommends each player i to play the best response; $\forall s_i, s_i' \in \mathcal{S}_i$:

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Coarse Correlated Equilibrium – a probability distribution over pure strategy profiles $p = \Delta(\mathcal{S})$ that **in expectation** recommends each player i to play the best response; $\forall s_i \in \mathcal{S}_i$:

$$\sum_{s' \in \mathcal{S}'} p(s') u_i(s') \geq \sum_{s' \in \mathcal{S}'} p(s') u_i(s_i, s_{-i}')$$

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(Coarse) Correlated equilibrium is often a result of a no-regret learning strategy in a game.

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- graphical games (action graph games)

Succinct Representations

Definition (Papadimitriou and Roughgarden, 2008)

A *succinct game* $G = (I, T, U)$ is defined, like all computational problems, in terms of a set of efficiently recognizable inputs I, and two polynomial algorithms T and U. For each $z \in I$, $T(z)$ returns a type, that is, an integer $n \geq 2$ (the number of players) and an n-tuple of integers $(t_1, \ldots, t_n)$, each at least 2 (the cardinalities of the strategy sets). If n and the t_p's are polynomially bounded in $|z|$, the game is said to be of *polynomial type*. Given any n-tuple of positive integers $s = (s_1, \ldots, s_n)$, with $s_p \leq t_p$ for all $p \leq n$, $U(z, p, s)$ returns an integer standing for the utility $u_p(s)$. The resulting game is denoted $G(z)$.

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For a *correlated equilibrium* σ it must hold:

$$\sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma(s_i, s_{-i}) \left(u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) \right) \geq 0 \quad \forall i \in \mathcal{N}, \forall s_i, s'_i \in \mathcal{S}_i$$

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Lemma:
For every $y \geq 0$, there is a product distribution σ such that $\sigma U^T y = 0$.

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$$[U\sigma_\ell^T]\alpha \geq 0 \qquad \alpha \geq 0$$

and α is a correlated equilibrium (a convex combination of product distributions over $\mathcal{S}$ that satisfies CE constraints).

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This approach does not generalize to finding some optimum correlated equilibrium. For example, maximizing the expected utility of players $(\max \sum_s u_s \sigma_s)$ and constraining σ to be a probability distribution $(\sum_s \sigma_s = 1)$ would lead to dual constraints

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$$1 + \sum_{i \in \mathcal{N}} |\mathcal{S}_i| \left(|\mathcal{S}_i| - 1 \right)$$

in polynomial time.

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1. Apply the ellipsoid method using the Purified Separation Oracle, a starting ball with radius of $R = u_{max}^{5\mathcal{N}^3}$ centered at 0, and stopping when the volume of the ellipsoid is below $v = \alpha_{\mathcal{N}} u_{max}^{-7N^5}$, where $\alpha_{\mathcal{N}}$ is the volume of the $\mathcal{N}$-dimensional unit ball.

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2. Form the matrix U' whose columns are $U_{s^{(1)},\ldots,s^{(L)}}$ generated by the separation oracle during the run of the ellipsoid method.
3. Find a feasible solution x' of the linear feasibility program

$$U'x' \geq 0,\ x' \geq 0,\ \mathbf{1}^\top x' = 1.$$

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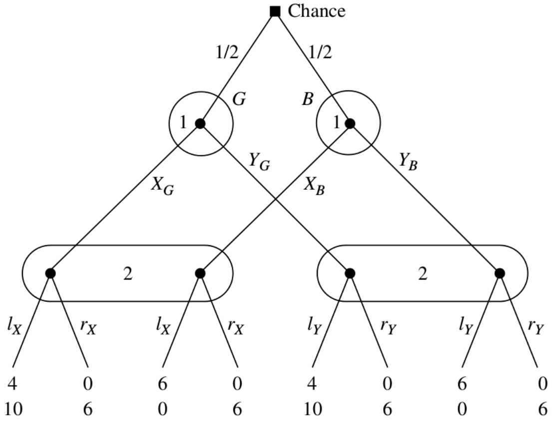
  - formally defined as *Extensive-Form Correlated Equilibrium* (EFCE)
  - computing one EFCE is computable in polynomial time
  - computing an optimal EFCE is NP-hard for almost all cases (two-player games with no chance is the exception)

Extensive-Form Correlated Equilibrium



Extensive-Form Correlated Equilibrium

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Representation of strategies in the two-player case: probability distribution over pairs of *relevant sequences*.

$$p(\emptyset, \emptyset) = 1; \quad 0 \leq p(\sigma_1, \sigma_2) \leq 1 \tag{1}$$

$$p(\sigma_i, \sigma_{-i}) = \sum_{a \in A(I)} p(\sigma_i a, \sigma_{-i}) \qquad \forall I \in \mathcal{I}_i, \sigma_i = \mathsf{seq}_i(I), \forall \sigma_{-i} \in rel(\sigma_i) \tag{2}$$

$$v(\sigma_{-i}) = \sum_{\sigma_i \in rel(\sigma_{-i})} p(\sigma_i, \sigma_{-i}) g_{-i}(\sigma_i, \sigma_{-i}) + \sum_{a \in A_{-i}(I)} v(\sigma_{-i} a) \quad \forall \sigma_{-i} \in \Sigma_{-i} \tag{3}$$

$$v(I, \sigma_{-i}) \geq \sum_{\sigma_i \in rel(\sigma_{-i})} p(\sigma_i, \sigma_{-i}) g_{-i}(\sigma_i, \mathsf{seq}_{-i}(I)a) + \sum_{I' \in \mathcal{I}_{-i};\ \mathsf{seq}_{-i}(I') = \sigma_{-i}(I)a} v(I', \sigma_{-i}) \tag{4}$$

$$v(\mathsf{seq}_{-i}(I)a) = v(I, \mathsf{seq}_{-i}(I)a) \qquad \forall I \in \mathcal{I}_{-i}, \forall a \in A(I) \tag{5}$$

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$$V_i^\pi(h) = \sum_a \pi(h,a) Q_i^\pi(h,a,a)$$

$$Q_i^\pi(h,a,a') = R(s(h),a') + \gamma \sum_{s'} P(s'|s(h),a') V_i^\pi(\langle h,a,a',s'\rangle)$$

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$$\forall (h,i,a_i,a_i') \qquad Q_i^{\pi}(h,a_i,a_i) \geq Q_i^{\pi}(h,a_i,a_i')$$

Computing Correlated Equilibrium in Stochastic Games

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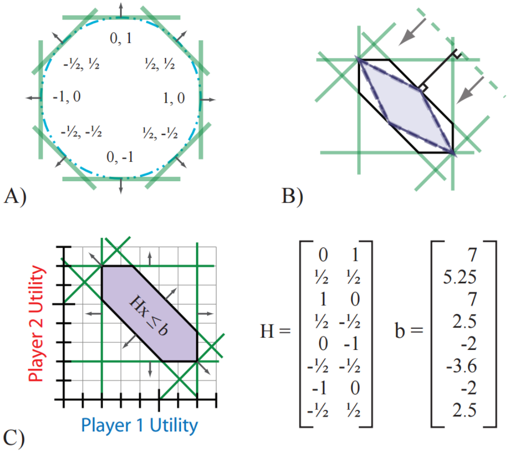
This gives us a compact approximate representation (it is sufficient to remember the offset) that further simplifies value backup functions – this generally leads to Minkowski sum of convex sets.

Computing Correlated Equilibrium in Stochastic Games



$$H = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 5.25 \\ 7 \\ 2.5 \\ -2 \\ -3.6 \\ -2 \\ 2.5 \end{bmatrix}$$ [1]

[1] Figure from [3]

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Alternatively, we may require subgame perfection – i.e., even after a deviation the players play rationally [4].

Atomic Congestion Games

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We have n players, set of edges E, strategies for each player are *paths* in the network ($\mathcal{S}$), and there is a congestion function $c_e : \{0, 1, \ldots, n\} \to \mathbb{Z}^+$. When all players choose their strategy path $s_i \in \mathcal{S}_i$ we have the load of edge e, $\ell_s(e) = |\{s_i : e \in s_i\}|$ and $u_i = -\sum_{e \in s_i} c_e(\ell_s(e))$.

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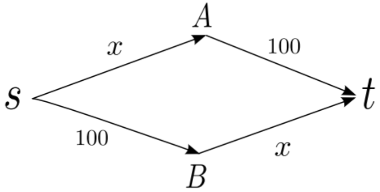

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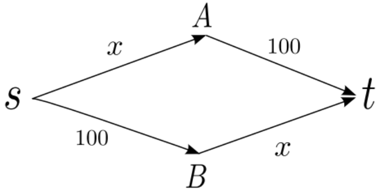

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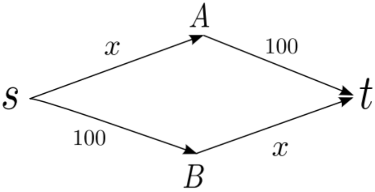

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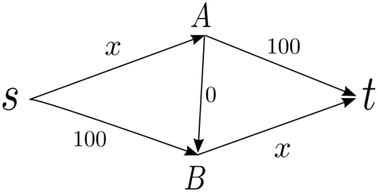

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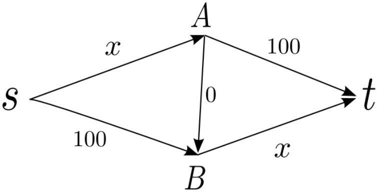

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Function ϕ attains a minimum (that must exist) at a Nash equilibrium. $\square$

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Many works study *Price of Anarchy* (or other) concepts in such games.

Generalization to Potential Games

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Theorem (shortened [5])

Any PLS problem can be reduced in polynomial time to a general potential game.

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a

	C	D
C	$-1, -1$	$-6, \ 0$
D	$0, -6$	$-4, -4$

b

	C	D
C	1	2
D	2	4

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- for each player $i \in \mathcal{N}$ and strategy profile $s = (s_1, \ldots, s_n)$, the utility of player i is

$$u_i(s) = \sum_{\forall j \in \mathcal{N} : (i,j) \in \mathcal{E}} u^{ij}(s_i, s_j)$$

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Examples: coordination game among agents, games among agents in a network

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$$
\sum_{i \in \mathcal{N}} w_i \geq \sum_{i \in \mathcal{N}} \max_{s \in \mathcal{S}_i} u_i(s, x_{-i}) = \max_{x_i \in \Delta(\mathcal{S}_i)} \sum_{i \in \mathcal{N}} u_i(s, x_{-i}) \geq 0
$$

Setting $w_i = \max_{s \in \mathcal{S}_i} u_i(s, x^*_{-i})$, where x^* is a NE is a feasible solution (and vice versa).

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