2. The most probable realisation of a GRF

$S = \{S_i \in K \mid i \in V\}$ is a K-valued Gibbs random field w.r.t. an undirected graph (V, E).

$$p(s) = \frac{1}{Z(u)} e^{\langle u, \varphi(s)\rangle} = \frac{1}{Z(u)} \exp \sum_{ij \in E}' u_{ij}(s_i, s_j)$$

Task: Find the most probable realisation(s) $s^* \in K^V$, i.e.

$$s^* \in \operatorname*{argmax}_{s \in K^V} \frac{1}{Z(u)} e^{\langle u, \varphi(s)\rangle} = \operatorname*{argmax}_{s \in K^V} \sum_{ij \in E}' u_{ij}(s_i, s_j) \qquad (1)$$

Remarks

- The task is NP complete (e.g. reduce the max-clique task)
- The task is solvable in polynomial time if (V, E) is a tree
- The task ——— " ——— if all functions $u_{ij}: K^2 \to \mathbb{R}$ are supermodular w.r.t. some total ordering of K. □

Definition 1 Let K be a totally ordered finite set. A function $u: K^n \to \mathbb{R}$ is submodular if

$$u(s) + u(s') \geq u(s \vee s') + u(s \wedge s')$$

holds for any pair $s, s' \in K^n$. A function $u: K^n \to \mathbb{R}$ is supermodular if $-u$ is submodular. □

Remark 1 $s \vee s'$ and $s \wedge s'$ denote the element-wise maximum and minimum of the tuples $s, s' \in K^n$. □

In general, we have to rely on approximation algorithms. One option: Relax the discrete optimisation problem to a linear optimisation problem.

LP relaxation

We start from an upper bound of (1) and minimise it w.r.t. re-parametrisations.

$$M_* = \max_{s \in K^V} \sum_{ij \in E} u_{ij}(s_i, s_j) \leqslant \sum_{ij \in E} \max_{s_i, s_j \in K} u_{ij}(s_i, s_j)$$

$$M_* \leqslant \sum_{ij \in E} \max_{s_i, s_j \in K} \left[\varphi_{ij}(s_i) + u_{ij}(s_i, s_j) + \varphi_{ji}(s_j) \right] \to \min_{\varphi}$$

$$\text{s.t.} \quad \sum_{j \in N_i} \varphi_{ij}(s_i) = 0 \quad \forall i \in V, \ \forall s_i \in K$$

Make it a linear optimisation problem

$$\sum_{ij \in E} c_{ij} \to \min_{\varphi, c}$$

$$\text{s.t.} \quad c_{ij} - \varphi_{ij}(s_i) - \varphi_{ji}(s_j) \geqslant u_{ij}(s_i, s_j) \quad \forall ij \in E, \ \forall s_i, s_j \in K \tag{2a}$$

$$\sum_{j \in N_i} \varphi_{ij}(s_i) = 0 \quad \forall i \in V, \ \forall s_i \in K$$

Construct its dual problem

$$\sum_{ij \in E} \sum_{s_i, s_j \in K} u_{ij}(s_i, s_j) \lambda_{ij}(s_i, s_j) \to \max_{\lambda \geqslant 0}$$

$$\text{s.t.} \quad \sum_{s_i, s_j \in K} \lambda_{ij}(s_i, s_j) = 1 \quad \forall ij \in E$$

$$\sum_{s_j \in K} \lambda_{ij}(s_i, s_j) = \lambda_i(s_i) \quad \forall ij \in E, \ \forall s_i \in K$$

Remarks

- The $\lambda_{ij}: K^2 \to \mathbb{R}_+$ describe relaxed labellings (weights). They encode a labelling if they are integral, i.e. $\lambda_{ij}(s_i, s_j) = 0, 1$
- It might seem that the λ can be interpreted as marginal probabilities. This is not true

$$\lambda \in \operatorname{aff}\{\varphi(s) \mid s \in K^V\} \cap \mathbb{R}^n_+, \text{ whereas } \mu \in \operatorname{conv}\{\varphi(s) \mid s \in K^V\}$$

- If L_* is the optimal value of the LP (2) and M_* is the optimal value of (1), then $L_* \geq M_*$ and, in general, there is an integrality gap, i.e. $L_* > M_*$.

Theorem 1 If all functions u_{ij} in (1) are supermodular w.r.t. some total ordering of K, then there is no integrality gap between the optimal values of (1) and its LP relaxation (2). □

Proof (Idea)

- Let λ^* be an optimiser of (2b). Find the highest label with non-zero weight in each node $i \in V$

$$k_i^* = \max\{k \in K \mid \lambda_i^*(k) > 0\}$$

- Show, there is another optimiser $\tilde{\lambda}^*$ s.t.

$$\tilde{\lambda}^*_{ij}(k_i^*, k_j^*) > 0 \quad \forall\, ij \in E$$

- Conclude, that the labelling s^*: $s_i^* = k_i^* \quad \forall i \in V$ is optimal.