Assignment 1. Consider the task of finding the most probable sequence of (hidden) states for a (Hidden) Markov model on a chain.

a) Show that the Dynamic Programming approach applied for this task can be interpreted as an equivalent transformation (re-parametrisation) of the model.

b) Show that the transformed functions (potentials) encode an explicit description of all optimisers of the problem.

Assignment 2. Prove that a sum of submodular functions is submodular.

Assignment 3. Examine the following functions w.r.t. submodularity

a) \( f(k, k') = |k - k'| \), where \( k, k' \in \mathbb{Z} \).

b) \( f(k, k') = (k - k')^2 \), where \( k, k' \in \mathbb{Z} \).

c) \( f(k_1, \ldots, k_n) = \max_i k_i - \min_i k_i \), where \( k_i \in \mathbb{Z} \).

Assignment 4. Let \( K \) be a completely ordered finite set. We assume w.l.o.g. \( K = \{1, 2, \ldots, m\} \). For a function \( u : K \to \mathbb{R} \) define its discrete “derivative” by \( Du(k) = u(k + 1) - u(k) \).

a) Let \( u \) be a function \( u : K^2 \to \mathbb{R} \) and denote by \( D_1 \) and \( D_2 \) the discrete derivatives w.r.t. the first and second argument. Prove the following equality

\[
D_1 D_2 u(k_1, k_2) = u(k_1 + 1, k_2 + 1) + u(k_1, k_2) - u(k_1 + 1, k_2) - u(k_1, k_2 + 1).
\]

Conclude that all mixed derivatives \( D_1 D_2 u(k_1, k_2) \) of a submodular functions are non-positive.

b) Prove that the condition established in a) is necessary and sufficient for a function to be submodular. *Hint:* Start from the observation that the following equality holds for a function of one variable

\[
u(k + l) - u(k) = \sum_{i=k}^{k+l-1} Du(i)\]

and generalise it for functions of two variables.

c) Prove that any function \( u : K^2 \to \mathbb{R} \) can be represented as a sum of a submodular and a supermodular function. *Hint:* Consider the mixed derivative \( D_1 D_2 u(k_1, k_2) \), decompose it into its negative and positive part and “integrate” them back separately.

Assignment 5. Consider a GRF for binary valued labellings \( x : V \to \{0, 1\} \) of a graph \((V, E)\) given by

\[
p(x) = \frac{1}{Z} \exp \left[ \sum_{ij \in E} u_{ij}(x_i, x_j) + \sum_{i \in V} u_i(x_i) \right].
\]

Show that is is always possible to find an equivalent transformation (re-parametrisation)

\[
u_{ij} \to \tilde{u}_{ij}, \quad u_i \to \tilde{u}_i
\]
such that the new pairwise functions $\tilde{u}_{ij}$ have the form

$$\tilde{u}_{ij}(x_i, x_j) = \alpha_{ij}|x_i - x_j|$$

with some real numbers $\alpha_{ij} \in \mathbb{R}$.

**Assignment 6.** Transform the *Travelling Salesman Problem* into a $(\min, +)$-problem.

**Assignment 7.** Consider the language $L$ of all b/w images $x : D \to \{b, w\}$ containing an arbitrary number of non-overlapping and non-touching one pixel wide rectangular frames (see figure).

![Diagram of a b/w image with non-overlapping one pixel wide rectangular frames.]

**a)** Prove that $L$ is not expressible by a locally conjunctive predicate

$$x \in L \text{ if and only if } f(x) = \bigwedge_{c \in \mathcal{C}} f_c(x_c) = 1$$

with predicates $f_c$, defined on image fragments $x_c$, where the hyperedges $c \subset D$ have bounded size $|c| \leq m < |D|$.

**b)** Show that $L$ can be expressed by introducing a field $s : D \to K$ of non-terminal symbols, a locally conjunctive predicate on them and pixel-wise predicates $g$ relating the non-terminal and terminal symbol in each pixel

$$x \in L \text{ if and only if } \bigvee_{s \in K^D} \left[ \bigwedge_{c \in \mathcal{C}} f_c(s_c) \land \bigwedge_{i \in D} g(x_i, s_i) \right] = 1$$

Find a suitable structure $\mathcal{C}$, an alphabet of non-terminal symbols $K$ and predicates $f_c, g$. 