Two Planar Homographies

3D Computer Vision – Lab Session Task

(CTU FEE subjects B4M33TDV, BE4M33TDV, XP33VID)

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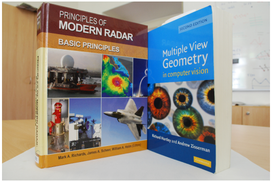
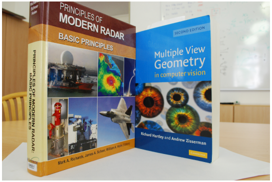
Task Outline

Task: **Find two dominant planar homographies between two selected views of the scene using the homology constraint.**

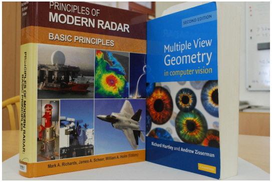
1. Select at least one pair of images from the set below: books_1.png, books_2.png and books_3.png.
2. Download the corresponding keypoints: books_u1.txt, books_u2.txt and books_u3.txt; and the *tentative* correspondences (zero-based indexing) between the selected pair of views: books_m12.txt, books_m13.txt and books_m23.txt.
3. Use robust estimation (RANSAC) to find the two dominant (book covers) homographies fulfilling the homology constraint.
4. Show outliers and inliers of both homographies (in different colours) – as a needle map – and plot the common line.



View 1

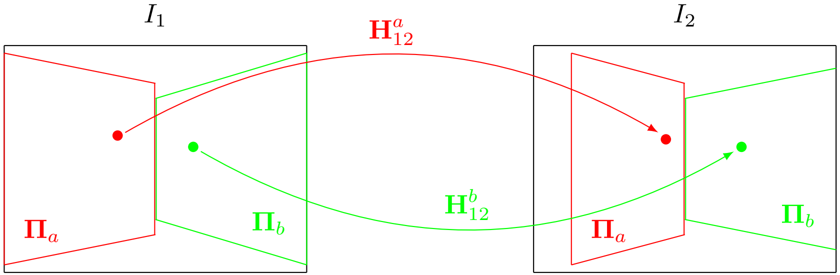


View 2



View 3

Let us have two views (images) of one scene with two dominant planes. We are searching for two homographies $\mathbf{H}_{12}^{a}$ and $\mathbf{H}_{12}^{b}$ that transform points from the first image onto the corresponding points in the second image for planes $\mathbf{\Pi}_a$ and $\mathbf{\Pi}_b$, respectively.



Sparse Correspondences

The sparse correspondences for provided images has been precomputed and they are available. Note, that the correspondences are tentative, so they may contain mismatches.

The correspondences are stored in several files:

- detected image keypoints (here `books_u<id>.txt`)
- 0-based indices of corresponding keypoints (here `books_m<i1><i2>.txt`)

Example: Working with correspondences

Keypoints in Image 1

```
5.3 1613.4
7.0  364.8
9.5 1522.3
9.9  585.1
10.9  571.7  <---
11.2  578.6
11.3  666.1
    ...
```

Keypoints in Image 2

```
6.3 1749.0
8.4 1753.3
8.9  497.9
10.4  540.9
11.0  683.2
11.0  687.8
11.1  589.8
11.3  583.4  <---
12.1 1212.6
12.2  949.3
    ...
```

Correspondences

```
4 7 <---
5 18285
11 27631
  ...
```

Transformation between two projective planes

vector form   elementwise form

$$\lambda \mathbf{x}_2 = \mathbf{H}\mathbf{x}_1 \qquad \lambda \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}$$

- bijection, $\mathbf{H}$ is regular, invertible
- $\lambda \neq 0$, any nonzero multiple of $\mathbf{H}$ represents the same transformation
- works for ideal points ($w_1 = 0$, $w_2 = 0$) as well
- in the affine plane (no ideal points) we can work with $w_1 = 1$, $w_2 = 1$ fixed

Notation of rows of $\mathbf{H}$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix}$$

i.e., $\mathbf{h}_i$ (without the $^\top$) is the i-th row of the matrix transposed to the column vector.

- Express particular elements of the vector $\underline{\mathbf{x}}_2$

vector form				elementwise form		
λu_2	$=$	$\mathbf{h}_1^\top \underline{\mathbf{x}}_1$	(1)	λu_2	$=$	$h_{11}u_1 + h_{12}v_1 + h_{13}w_1$
λv_2	$=$	$\mathbf{h}_2^\top \underline{\mathbf{x}}_1$	(2)	λv_2	$=$	$h_{21}u_1 + h_{22}v_1 + h_{23}w_1$
λw_2	$=$	$\mathbf{h}_3^\top \underline{\mathbf{x}}_1$	(3)	λw_2	$=$	$h_{31}u_1 + h_{32}v_1 + h_{33}w_1$

- Eliminate nonzero λ – multiply sides of eq. (1) and (2) by swapped sides of eq. (3) divided by λ, i.e., multiply the right side of (1) and (2) by w_2 and the left side of (1) and (2) by $\mathbf{h}_3^\top \underline{\mathbf{x}}_1 / \lambda$.
  Note: we do not divide by any of the coordinate entries (some can be zero).

$$u_2 \mathbf{h}_3^\top \underline{\mathbf{x}}_1 = w_2 \mathbf{h}_1^\top \underline{\mathbf{x}}_1 \qquad u_2(h_{31}u_1 + h_{32}v_1 + h_{33}w_1) = w_2(h_{11}u_1 + h_{12}v_1 + h_{13}w_1)$$
$$v_2 \mathbf{h}_3^\top \underline{\mathbf{x}}_1 = w_2 \mathbf{h}_2^\top \underline{\mathbf{x}}_1 \qquad v_2(h_{31}u_1 + h_{32}v_1 + h_{33}w_1) = w_2(h_{21}u_1 + h_{22}v_1 + h_{23}w_1)$$

- Some manipulation – to the homogeneous form, 'transpose' dot products

$$\begin{array}{lll} w_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_1 & & -u_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_3 = 0 \\ & w_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_2 & -v_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_3 = 0 \end{array}$$

$$\begin{array}{lll} w_2u_1h_{11} + w_2v_1h_{12} + w_2w_1h_{13} & & -u_2u_1h_{31} - u_2v_1h_{32} - u_2w_1h_{33} = 0 \\ & w_2u_1h_{21} + w_2v_1h_{22} + w_2w_1h_{23} & -v_2u_1h_{31} - v_2v_1h_{32} - v_2w_1h_{33} = 0 \end{array}$$

Homography Estimation from Known Correspondences (2/3)

- Matrix representation – collect all known terms (point coordinates from the i-th corresponding pair of points) to a matrix $\mathbf{A}_i$ and all unknowns to a vector $\mathbf{h}$

$$\underbrace{\begin{bmatrix} w_2\underline{\mathbf{x}}_1^\top & \mathbf{0}^\top & -u_2\underline{\mathbf{x}}_1^\top \\ \mathbf{0}^\top & w_2\underline{\mathbf{x}}_1^\top & -v_2\underline{\mathbf{x}}_1^\top \end{bmatrix}}_{\mathbf{A}_i} \underbrace{\begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix}}_{\mathbf{h}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} w_2u_1 & w_2v_1 & w_2w_1 & 0 & 0 & 0 & -u_2u_1 & -u_2v_1 & -u_2w_1 \\ 0 & 0 & 0 & w_2u_1 & w_2v_1 & w_2w_1 & -v_2u_1 & -v_2v_1 & -v_2w_1 \end{bmatrix}}_{\mathbf{A}_i} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{h}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\mathbf{A}_i$ – 2 equations, $\mathbf{h}$ – 8 unknowns (up to scalar multiple) $\longrightarrow$ 4 correspondences needed, stacked to 8×9 matrix $\mathbf{A}$. Then $\mathbf{h}$ is a solution of linear homogeneous system, i.e. the basis vector of the null space of $\mathbf{A}$, and $\mathbf{H}$ is composed from $\mathbf{h}$.

$$\mathbf{A} = \begin{bmatrix} \vdots \\ \mathbf{A_i} \\ \vdots \end{bmatrix} \qquad \mathbf{A}\mathbf{h} = \mathbf{0}$$

Hint: Python

```
_, _, vh = np.linalg.svd(A)
H = vh[-1, :].reshape((3,3))
```

Note: Assume we work with points from the affine plane only with coordinates normalized to $w_1 = 1$, $w_2 = 1$.

$$\mathbf{A} = \begin{bmatrix} & \vdots & \\ \underline{\mathbf{x}}_1^\top & \mathbf{0}^\top & -u_2 \underline{\mathbf{x}}_1^\top \\ \mathbf{0}^\top & \underline{\mathbf{x}}_1^\top & -v_2 \underline{\mathbf{x}}_1^\top \\ & \vdots & \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} & & & & \vdots & & & & \\ u_1 & v_1 & 1 & 0 & 0 & 0 & -u_2 u_1 & -u_2 v_1 & -u_2 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -v_2 u_1 & -v_2 v_1 & -v_2 \\ & & & & \vdots & & & & \end{bmatrix}$$

Try using the RANSAC (or MLESAC) scheme for robust estimation of the first homography $\mathbf{H}^a_{12}$. I.e., sample four random correspondences and create the model (using the method from slides 5-8). Select a reasonable threshold in pixels and use the *Euclidean* distance between the corresponding points:

$$\varepsilon_{\mathbf{H}}(\mathbf{x}_1, \mathbf{x}_2) = ||\mathbf{x}_2 - \mathbf{x}_{12}||\,, \text{ where } \quad \underline{\mathbf{x}}_{12} = \mathbf{H}\,\underline{\mathbf{x}}_1 \quad \text{and} \quad I_1 \ni \mathbf{x}_1 \sim \mathbf{x}_2 \in I_2,$$

to find the inliers and meassure the quality of such homography. Verify the best homography using the needle map visualization of its inlier correspondences.

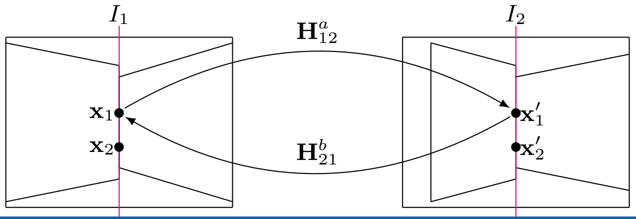
- Consider a 2D common line between the two 3D planes (book covers, in our scenario), as depicted in the figure below. It must hold for any two distinct affine points $\mathbf{x}_1$ and $\mathbf{x}_2$ on this line that the two homographies $\mathbf{H}^a_{12}$ and $\mathbf{H}^b_{12}$ perform the same transformation. Hence, it holds that:

$$\lambda_1 \underline{\mathbf{x}}_1 = \mathbf{H}^b_{21}\mathbf{H}^a_{12}\underline{\mathbf{x}}_1 \quad \text{and} \quad \lambda_2 \underline{\mathbf{x}}_2 = \mathbf{H}^b_{21}\mathbf{H}^a_{12}\underline{\mathbf{x}}_2.$$

- For any of their linear combinations: $\underline{\mathbf{x}}_3 = \alpha\underline{\mathbf{x}}_1 + \beta\underline{\mathbf{x}}_2$, it must hold that:

$$\lambda_3 \underline{\mathbf{x}}_3 = \mathbf{H}^b_{21}\mathbf{H}^a_{12}\underline{\mathbf{x}}_3 = \alpha\mathbf{H}^b_{21}\mathbf{H}^a_{12}\underline{\mathbf{x}}_1 + \beta\mathbf{H}^b_{21}\mathbf{H}^a_{12}\underline{\mathbf{x}}_2 = \alpha\lambda_1\underline{\mathbf{x}}_1 + \beta\lambda_2\underline{\mathbf{x}}_2.$$

  This requires us to set $\lambda_1 = \lambda_2 = \lambda_3$, which means that this special homography $\mathbf{H} = \mathbf{H}^b_{21}\mathbf{H}^a_{12}$ needs to have two eigenvectors with the same eigenvalue.
  - This would not be usually satisfied between two random homographies found separately. Hence, we will look for a so-called homology instead.



- Homography between two images does not depend on the world coordinate system choice

(chosen at the first camera matrix $\mathbf{P_1}$)

- Two cameras $\mathbf{P}_1$ and $\mathbf{P}_2$ observing a plane $\mathbf{\Pi}$

$$\mathbf{P}_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \quad \mathbf{P}_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \quad \underline{\mathbf{\Pi}}^\top = \begin{bmatrix} \mathbf{n}^\top & d \end{bmatrix}$$

1. Reconstruct $\underline{\mathbf{X}}$ constrained by the plane $\underline{\mathbf{\Pi}}$ from $\underline{\mathbf{u}}_1$
   (projection equation augmented by plane constraint row $\underline{\mathbf{\Pi}}^\top \underline{\mathbf{X}} = 0$ to obtain 4×4 invertible matrix)

$$\lambda_1 \begin{bmatrix} \underline{\mathbf{u}}_1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} \\ \mathbf{n}^\top & d \end{bmatrix}}_{\mathbf{M}} \underline{\mathbf{X}}$$

$$\underline{\mathbf{X}} = \lambda_1 \underbrace{\begin{bmatrix} \mathbf{K}_1^{-1} & \mathbf{0} \\ \frac{-\mathbf{n}^\top \mathbf{K}_1^{-1}}{d} & \frac{1}{d} \end{bmatrix}}_{\mathbf{M}^{-1}} \begin{bmatrix} \underline{\mathbf{u}}_1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} \mathbf{K}_1^{-1} \\ \frac{-\mathbf{n}^\top \mathbf{K}_1^{-1}}{d} \end{bmatrix} \underline{\mathbf{u}}_1$$

2. Project $\underline{\mathbf{X}}$ to the second camera to obtain $\mathbf{H}$

$$\lambda_2 \underline{\mathbf{u}}_2 = \mathbf{P_2} \underline{\mathbf{X}} = \lambda_1 \mathbf{K_2} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{K_1}^{-1} \\ \frac{-\mathbf{n}^\top \mathbf{K}_1^{-1}}{d} \end{bmatrix} \underline{\mathbf{u}}_1 = \lambda_1 \underbrace{\mathbf{K}_2 \left(\mathbf{R} - \mathbf{t}\mathbf{n}^\top / d\right) \mathbf{K}_1^{-1}}_{\mathbf{H}_{12}} \underline{\mathbf{u}}_1$$

3. Change of coordinate frame - apply $\mathbf{T}$ (4×4): $\underline{\mathbf{X}}' = \mathbf{T}\underline{\mathbf{X}}$, $\mathbf{P}' = \mathbf{P}\mathbf{T}^{-1}$, $\underline{\boldsymbol{\Pi}}'^\top = \underline{\boldsymbol{\Pi}}\mathbf{T}^{-1}$

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$\mathbf{P}_1' = \mathbf{K}_1 \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \end{bmatrix} \quad \mathbf{P}_2' = \mathbf{K}_2 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \quad \underline{\boldsymbol{\Pi}}'^\top = \begin{bmatrix} \underbrace{\mathbf{n}^\top \mathbf{R}^\top}_{\mathbf{n}'^\top} & \underbrace{-\mathbf{n}^\top \mathbf{R}^\top \mathbf{t} + d}_{d'} \end{bmatrix}$$

$\mathbf{n}'$, d' are parameters of the plane w.r.t. coordinate system of the second camera

4. Use eq. from Page 11, substitute $\mathbf{P}_1 \leftarrow \mathbf{P}_2'$, $\mathbf{P}_2 \leftarrow \mathbf{P}_1'$, $\underline{\boldsymbol{\Pi}} \leftarrow \underline{\boldsymbol{\Pi}}'$

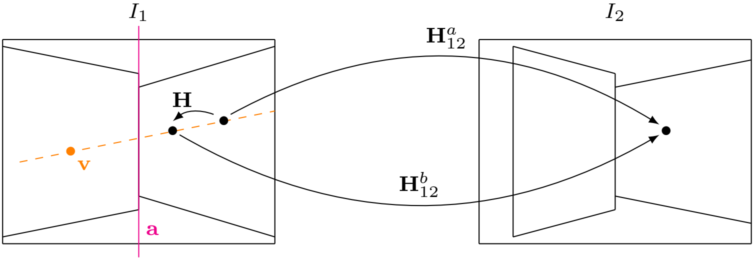
$$\mathbf{H}_{21} = \mathbf{K}_1 \left(\mathbf{R}^\top + \frac{\mathbf{R}^\top \mathbf{t} \mathbf{n}^\top \mathbf{R}^\top}{d - \mathbf{n}^\top \mathbf{R}^\top \mathbf{t}} \right) \mathbf{K}_2^{-1} = \mathbf{K}_1 \left(\mathbf{R}^\top + \frac{\mathbf{R}^\top \mathbf{t} \mathbf{n}'^\top}{d'} \right) \mathbf{K}_2^{-1}$$

(can be easily verified that $\mathbf{H}_{21} = \mathbf{H}_{12}^{-1}$ from Page 11)

- A pair of cameras $\mathbf{K_1}[\mathbf{I}|\mathbf{0}]$, $\mathbf{K_2}[\mathbf{R}|\mathbf{t}]$ observes a pair of planes $\underline{\boldsymbol{\Pi}}_a$, $\underline{\boldsymbol{\Pi}}_b$ $\to$ $\mathbf{H}^a_{12}$, $\mathbf{H}^b_{12}$
- Consider a composed homography 'there-and-back': $1 \to 2$ via $\underline{\boldsymbol{\Pi}}_a$ and $2 \to 1$ via $\underline{\boldsymbol{\Pi}}_b$

$$\begin{aligned}\mathbf{H} &= \mathbf{H}^b_{21}\mathbf{H}^a_{12} = \mathbf{K}_1(\mathbf{R}^\top + \mathbf{R}^\top\mathbf{t}\mathbf{n}_b'^\top/d_b')\mathbf{K_2}^{-1}\,\mathbf{K_2}(\mathbf{R} - \mathbf{t}\mathbf{n}_a^\top/d_a)\mathbf{K}_1^{-1} = \\ &= \mathbf{I} + \underbrace{(-\mathbf{K_1}\mathbf{R}^\top\mathbf{t})}_{\mathbf{v}}\underbrace{\left(\frac{\mathbf{n}_a^\top}{d_a} - \frac{\mathbf{n}_b'^\top\mathbf{R}}{d_b'} + \frac{\mathbf{n}_b'^\top\mathbf{t}\mathbf{n}_a^\top}{d_a\,d_b'}\right)\mathbf{K_1}^{-1}}_{\mathbf{a}^\top} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top\end{aligned}$$

Planar homology $\mathbf{H}$ with vertex $\mathbf{v}$, axis $\mathbf{a}$, eigen value $\mu = 1 + \mathbf{a}^\top\mathbf{v}$: $\quad \mathbf{H} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$.



- A pair of cameras $\mathbf{K_1}[\mathbf{I}|\mathbf{0}]$, $\mathbf{K_2}[\mathbf{R}|\mathbf{t}]$ observes a pair of planes $\underline{\boldsymbol{\Pi}}_a$, $\underline{\boldsymbol{\Pi}}_b \rightarrow \mathbf{H}_{12}^a$, $\mathbf{H}_{12}^b$
- Consider a composed homography 'there-and-back': $1 \rightarrow 2$ via $\underline{\boldsymbol{\Pi}}_a$ and $2 \rightarrow 1$ via $\underline{\boldsymbol{\Pi}}_b$

$$\begin{aligned}\mathbf{H} &= \mathbf{H}_{21}^b \mathbf{H}_{12}^a = \mathbf{K}_1(\mathbf{R}^\top + \mathbf{R}^\top \mathbf{t}\mathbf{n}_b'^\top / d_b')\mathbf{K_2}^{-1}\, \mathbf{K_2}(\mathbf{R} - \mathbf{t}\mathbf{n}_a^\top / d_a)\mathbf{K}_1^{-1} = \\ &= \mathbf{I} + \underbrace{(-\mathbf{K_1}\mathbf{R}^\top \mathbf{t})}_{\mathbf{v}} \underbrace{\left(\frac{\mathbf{n}_a^\top}{d_a} - \frac{\mathbf{n}_b'^\top \mathbf{R}}{d_b'} + \frac{\mathbf{n}_b'^\top \mathbf{t}\mathbf{n}_a^\top}{d_a\, d_b'}\right)\mathbf{K_1}^{-1}}_{\mathbf{a}^\top} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top\end{aligned}$$

Planar homology $\mathbf{H}$ with vertex $\mathbf{v}$, axis $\mathbf{a}$, eigen value $\mu = 1 + \mathbf{a}^\top \mathbf{v}$: $\qquad \mathbf{H} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$

$\mathbf{H}\mathbf{v} = \mathbf{v} + \mathbf{v}\mathbf{a}^\top \mathbf{v} = (1 + \mathbf{a}^\top \mathbf{v})\mathbf{v} = \mu\mathbf{v}$ $\quad \mathbf{v}$ is eigenvector of $\mathbf{H}$ corresponding to eigenvalue μ, as a planar point it is fixed w.r.t. the transformation

$\mathbf{a}^\top \underline{\mathbf{x}} = 0 \Longrightarrow \mathbf{H}\underline{\mathbf{x}} = \underline{\mathbf{x}} + \mathbf{v}\mathbf{a}^\top \underline{\mathbf{x}} = \underline{\mathbf{x}}$ $\quad \mathbf{a}$ is a line of fixed points - 2D eigenspace of $\mathbf{H}$ with double eigenvalue 1

- $\mathbf{a}$ and $\mathbf{v}$ represent homogeneous image entities but their scale matters in $\mathbf{I} + \mathbf{v}\mathbf{a}^\top$
- the point $\mathbf{v}$ and all points on a line $\mathbf{a}$ in one image are mapped to the second image to points same for both $\mathbf{H}_{12}^a$ and $\mathbf{H}_{12}^b$: $\qquad \mathbf{H}_{12}^a \mathbf{v} \simeq \mathbf{H}_{12}^b \mathbf{v} \qquad (\mathbf{H}_{12}^a)^{-\top}\mathbf{a} \simeq (\mathbf{H}_{12}^b)^{-\top}\mathbf{a}$
- $\mathbf{a}$ is the image of the common line of the planes
- $\mathbf{v} = -\mathbf{K_1}\mathbf{R}^\top \mathbf{t}$ is the so called *epipole* in the first image
- eigenvectors of $\mathbf{H}$ are $(\mathbf{v}, \mathbf{x}_1, \mathbf{x}_2)$ corresponding to eigenvalues $(\mu, 1, 1)$, and $\mathbf{a} = \mathbf{x}_1 \times \mathbf{x}_2$
  (when the matrix is multiplied by an unknown scale λ, the eigenvalues become $(\lambda\mu, \lambda, \lambda)$)

Let us assume, we have found the first dominant homography $\mathbf{H}_{12}^a$ from I_1 to I_2. We would like to find the second homography $\mathbf{H}_{12}^b$ s.t.:

$$\mathbf{H}_{12}^b = \mathbf{H}_{12}^a \mathbf{H}^{-1},$$

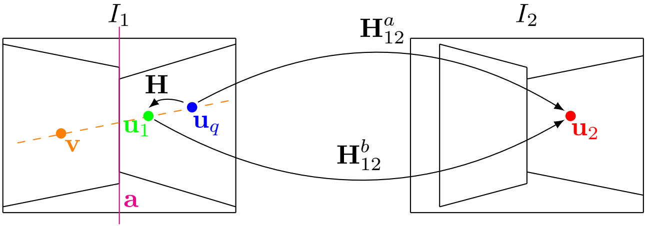
where $\mathbf{H} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$ is a homology. The situation is illustrated in the picture below.

For the puropose of our following derivations, we denote:

$$\mathbf{u} \overset{\text{def.}}{=} \mathbf{u}_q \quad \text{and} \quad \mathbf{u}' \overset{\text{def.}}{=} \mathbf{u}_1,$$

where $\lambda \underline{\mathbf{u}}_q = \mathbf{H}_{21}^a \underline{\mathbf{u}}_2$. We are then searching for a homology that satisfies:

$$\lambda \underline{\mathbf{u}}' = \mathbf{H} \underline{\mathbf{u}}.$$



1. A point $\underline{\mathbf{u}}$ and its image $\lambda\underline{\mathbf{u}}' = \mathbf{H}\underline{\mathbf{u}}$ define a common line $\underline{\mathbf{k}}$ (if not lying on $\mathbf{a}$ or $\mathbf{v}$). Since $\mathbf{a}^\top\underline{\mathbf{u}} \neq 0$, the vertex must lie on this line as well:

$$\underline{\mathbf{k}}^\top\underline{\mathbf{u}} = 0\,, \quad \underline{\mathbf{k}}^\top\mathbf{H}\mathbf{u} = 0$$
$$\underline{\mathbf{k}}^\top\left(\underline{\mathbf{u}} + \mathbf{v}\mathbf{a}^\top\underline{\mathbf{u}}\right) = 0 \quad\Longrightarrow\quad \underline{\mathbf{k}}^\top\mathbf{v}\mathbf{a}^\top\underline{\mathbf{u}} = 0 \quad\Longrightarrow\quad \underline{\mathbf{k}}^\top\mathbf{v} = 0$$

The vertex is estimated from two correspondences: $\mathbf{v} = (\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_1') \times (\underline{\mathbf{u}}_2 \times \underline{\mathbf{u}}_2')$

2. When the vertex $\mathbf{v}$ is known, a linear non-homogeneous system can be used for computing $\mathbf{a}$. Let $\mathbf{H} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$, $\underline{\mathbf{u}}_i^\top = [x_i, y_i, w_i]$, $\underline{\mathbf{u}}_i'^\top = [x_i', y_i', w_i']$, $\mathbf{v}^\top = [x_v, y_v, w_v]$.

$$\lambda\begin{bmatrix} x_i' \\ y_i' \\ w_i' \end{bmatrix} = \underline{\mathbf{u}}_i + \mathbf{v}\mathbf{a}^\top\underline{\mathbf{u}}_i = \underline{\mathbf{u}}_i + \mathbf{v}\underline{\mathbf{u}}_i^\top\mathbf{a} = \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix} + \begin{bmatrix} x_v\,\underline{\mathbf{u}}_i^\top \\ y_v\,\underline{\mathbf{u}}_i^\top \\ w_v\,\underline{\mathbf{u}}_i^\top \end{bmatrix}\mathbf{a}$$

$$\begin{aligned} x_i'(w_i + w_v\underline{\mathbf{u}}_i^\top\mathbf{a}) &= w_i'x_i + w_i'x_v\underline{\mathbf{u}}_i^\top\mathbf{a} \\ y_i'(w_i + w_v\underline{\mathbf{u}}_i^\top\mathbf{a}) &= w_i'y_i + w_i'y_v\underline{\mathbf{u}}_i^\top\mathbf{a} \end{aligned} \qquad \begin{aligned} (x_i'w_v - w_i'x_v)\underline{\mathbf{u}}_i^\top\mathbf{a} &= x_iw_i' - x_i'w_i \\ (y_i'w_v - w_i'y_v)\underline{\mathbf{u}}_i^\top\mathbf{a} &= y_iw_i' - y_i'w_i \end{aligned}$$

The left sides of these eqs. consists of the same vector $\underline{\mathbf{u}}_i^\top$ multiplied by a scalar, i.e., they are linearly dependent, and only one eq. can be used → three points needed.

$$\mathbf{A} = \begin{bmatrix} (x_1'w_v - w_1'x_v)\underline{\mathbf{u}}_1^\top \\ (x_2'w_v - w_2'x_v)\underline{\mathbf{u}}_2^\top \\ (x_3'w_v - w_3'x_v)\underline{\mathbf{u}}_3^\top \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_1w_1' - w_1x_1' \\ x_2w_2' - w_2x_2' \\ x_3w_3' - w_3x_3' \end{bmatrix} \qquad \mathbf{a} = \mathbf{A}^{-1}\mathbf{b}$$

There are many methods to estimate multiple geometric models (homographies in this case) from given data. Two simplest possibilities are:

- Sequential RANSAC
  1. Estimate the first homography $\mathbf{H}^a$ using standard RANSAC scheme for a single homography (see slide 9). Typically the most dominant one is found.
  2. Remove inliers of the first found homography $\mathbf{H}^a$ from the data.
  3. Estimate the second homography $\mathbf{H}^b$ using RANSAC on the remaining data, such that the common homology constraint is enforced (3 correspondences needed, see slides 13–16)
- Multi RANSAC with conditional sampling
  - Estimate both homographies simultaneously in RANSAC scheme by sampling 7-tuple of correspondences.
  - In every sample:
    1. Draw 4 random correspondences and estimate $\mathbf{H}^a$ (see slides 5–8)
    2. Remove inliers of $\mathbf{H}^a$ from the data set
    3. Draw 3 random correspondences and estimate $\mathbf{H}^b$ with homology constraint (see slides 13–16)
    4. Verify support of both homographies, accept the best one

Use either the Sequential RANSAC or Multi RANSAC to estimate the two dominant homographies with homology constraint. Show outliers and inliers of both homographies (in different colours) – as a needle map – and plot the common line.