Question 1. (2 points)

Consider a version-space agent whose initial hypothesis class $\mathcal{H}_1$ contains all non-contradictory conjunctions on 3 propositional variables.

1. Determine $|\mathcal{H}_1|$.

   Answer:

   $3^3 = 27$ (Each of the 3 variables may be absent, positive, or negative in the conjunction.)

2. Give an upper bound on $|\mathcal{H}_2|$ given that $r_2 = -1$.

   Answer:

   A version-space agent decides by majority vote so at least 14 hypotheses in $\mathcal{H}$ were inconsistent with x_1; these get deleted and at most 13 remain.

Question 2. (2 points)

Let $r_{\leq K}$ be a reward sequence of a standard agent and $h_{\leq K}$ be its sequence of hypotheses. Denote $M = \sum_{k=1}^{K} |r_k|$. Show that there is a hypothesis h retained for at least $\frac{K}{M+1}$ consecutive steps in $h_{\leq K}$, i.e.

$$h_{\leq K} = h_1, h_2, \ldots \underbrace{h, h, \ldots h}_{\text{at least } \frac{K}{M+1} \text{ times}}, \ldots h_K$$

Answer:

A standard agent changes its hypothesis ($h_{k+1} \neq h_k$) if and only if a mistake is made ($r_k = -1$). The number of mistakes up to time K is M so there are at most $M+1$ interleaving subsequences of unchanged hypotheses in $h_{\leq K}$ (there are exactly M of them if the M's mistake is made at time K, otherwise $M+1$). If each of the at most $M+1$ subsequences is shorter than $\frac{K}{M+1}$ then the length of the sequence is K less than $\frac{K}{M+1} \cdot (M+1) = K$, which is a contradiction.

Question 3. (5 points)

Let an agent PAC-learn $\mathcal{C}$ from X. Show that for any target concept from $\mathcal{C}$ on X, an arbitrary distribution $P(x)$ on X and arbitrary numbers $0 < \epsilon, \delta < 1$ and $K \in \mathbb{N}$, the condition $\mathsf{err}(h_K) \leq \epsilon$ with probability at least $1 - \delta$ implies that h_K is consistent with all observations in $x_{<K}$.

Answer:

Assume for contradiction that h_K is not consistent with some x_k, $1 \leq k \leq K$. The distribution $P(x)$ and the number $\delta < 1$ are arbitrary, so set them so that $\prod_{i=1}^{K} P(x_i) > \delta$ (which can evidently be done). This implies that $P(x_k) > 0$. The number $\epsilon > 0$ is also arbitrary so set it so that $\epsilon < P(x_k)$. Since h_K is inconsistent with x_k, $\mathsf{err}(h_K) \geq P(x_k) > \epsilon$. This happens with probability $\prod_{i=1}^{K} P(x_i) > \delta$ which contradicts the assumption of the implication.

Question 4. (5 points)

Let X contain all real numbers from $[0; 1]$ which can be represented using 256 bits. Let $\mathcal{H} = X$, and the decision policy given by a $h \in \mathcal{H}$ is

$$h(x) = 1 \text{ iff } x > h$$

Determine a k such that with probability at least 0.9, $\mathsf{err}(h) < 0.1$, where h is an arbitrary hypothesis from $\mathcal{H}$ consistent with k i.i.d. examples from X. Estimate it using:

1. $\ln |\mathcal{H}|$

2. $\mathsf{VC}(\mathcal{H})$

Answer:

1. The probability that some of the 2^{256} hypotheses is bad ($\mathsf{err}(h) > 0.1$) and still consistent with k i.i.d. observations is at most $2^{256}(1-0.1)^k = 2^{256}(0.9)^k$. We want this probability to be smaller than $1-0.9 = 0.1$:

$$2^{256}(0.9)^k < 0.1$$

i.e.

$$k > \log_{0.9} \frac{0.1}{2^{256}} \approx 1707 \text{ examples (smalllest such } k) \tag{1}$$

We can also use the general formula (see Lectures)

$$k > \frac{1}{\epsilon} \ln \frac{|\mathcal{H}|}{\delta} = \frac{1}{0.1} \ln \frac{2^{256}}{0.1} \approx 1798 \text{ examples (smalllest such } k)$$

which is slightly larger than (1) because the upper bound $(1-\epsilon)^k < e^{-\epsilon k} (\epsilon > 0)$ is used in the derivation of the formula.

2. Using $\mathsf{VC}(\mathcal{H})$. $\mathsf{VC}(\mathcal{H}) = 1$ because a single number from X can evidently be shattered (classified positively or negatively by hypotheses from $\mathcal{H}$) but two different numbers from X cannot be shattered: the smaller cannot be made positive while the larger is negative.

$$k > \max\left\{ \frac{4}{\epsilon} \lg \frac{2}{\delta}, \frac{8 \cdot \mathsf{VC}(\mathcal{H})}{\epsilon} \lg \frac{13}{\epsilon} \right\} \approx \max\{\, 173, 562 \,\} = 562 \text{ examples}$$