Markov Decision Process

+Assignment 2

Jan Mrkos

PUI Tutorial
Week 8

Outline

- Assignment 2
- Motivation
- MDP definition and examples
- MDP solution
- Value function calculation

Motivation

Any problems with stochastic outcomes

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.
- Purchase and production: how much to produce based on demand.

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.
- Purchase and production: how much to produce based on demand.
- Robotic navigation

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.
- Purchase and production: how much to produce based on demand.
- Robotic navigation

In addition, MDPs form a basis of many techniques in

[1] `https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes`

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.
- Purchase and production: how much to produce based on demand.
- Robotic navigation

In addition, MDPs form a basis of many techniques in

- Reinforcement Learning

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.
- Purchase and production: how much to produce based on demand.
- Robotic navigation

In addition, MDPs form a basis of many techniques in

- Reinforcement Learning
- Game theory (extensive form games)

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Motivation

Any problems with stochastic outcomes

- Dynamic pricing: deciding on prices for products based on demand, buying price, stock
- Maintenance and repair: when to replace/inspect based on age, condition, etc. [1]
- Agriculture: how much to plant based on weather and soil state.
- Purchase and production: how much to produce based on demand.
- Robotic navigation

In addition, MDPs form a basis of many techniques in

- Reinforcement Learning
- Game theory (extensive form games)

Important extension - Partial Observable MDPs

[1] https://stats.stackexchange.com/questions/145122/real-life-examples-of-markov-decision-processes

Markov Decision Process

Markovian:

Markov Decision Process

Markovian:

- Named after Andrey Markov (1856 - 1922)

Markov Decision Process

Markovian:

- Named after Andrey Markov (1856 - 1922)
- Memoryless, the next evolution of the systems depends ONLY on the current state, NOT on the sequence of events that lead to the state.

Markov Decision Process

Markovian:

- Named after Andrey Markov (1856 - 1922)
- Memoryless, the next evolution of the systems depends ONLY on the current state, NOT on the sequence of events that lead to the state.

Decision process:

Markov Decision Process

Markovian:

- Named after Andrey Markov (1856 - 1922)
- Memoryless, the next evolution of the systems depends ONLY on the current state, NOT on the sequence of events that lead to the state.

Decision process:

- You are expected to make a sequence of decision as responses to the changes in the environment.

Markov Decision Process

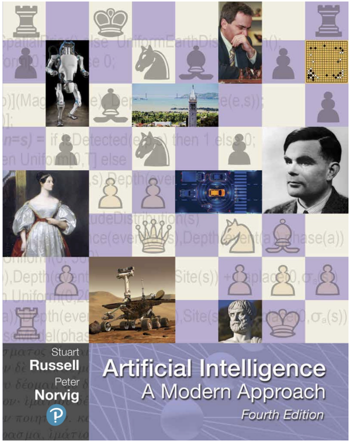
Markovian:

- Named after Andrey Markov (1856 - 1922)
- Memoryless, the next evolution of the systems depends ONLY on the current state, NOT on the sequence of events that lead to the state.

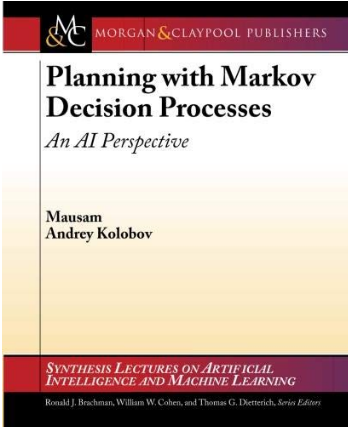
Decision process:

- You are expected to make a sequence of decision as responses to the changes in the environment.
- Plan vs. policy: "In planning, the problem is finding the plan. In MDP, the problem is executing the plan."

Resources



AI: A Modern Approach (Russel, Norwig)



Planning with Markov Decision Processes: An AI Perspective (Mausam, Kobolov)

Also, I have heard good things about the free `https://algorithmsbook.com/`.

Markov Decision Process

Tuple $\langle S, A, D, T, R \rangle$:

- S: finite set of states agent can find itself in
- A: finite set of action agent can perform
- D: finite set of timesteps
- T: transition function - transitions between states
- R: reward function - rewards obtained from transitions

Markov Decision Process

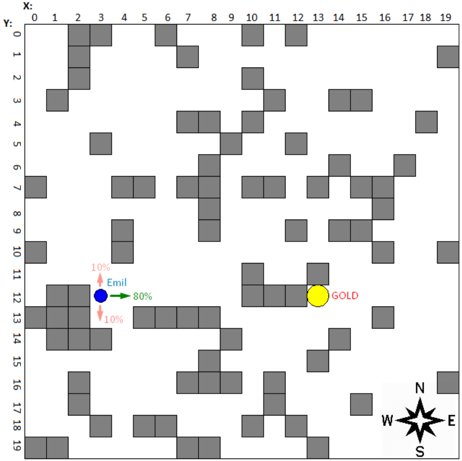
Tuple $\langle S, A, D, T, R \rangle$:

- S: finite set of states agent can find itself in
- A: finite set of action agent can perform
- D: finite set of timesteps
- T: transition function - transitions between states
- R: reward function - rewards obtained from transitions

⚠ Only one of many possible definitions!

Example: Emil in the gridworld

- S: Possible Emils positions
- A: Move directions
- D: Emil has e.g. 200 steps to find gold
- T: stochastic movement, e.g. 10% to move to the side of selected action
- R: e.g. +100 for finding gold, -1 for each move



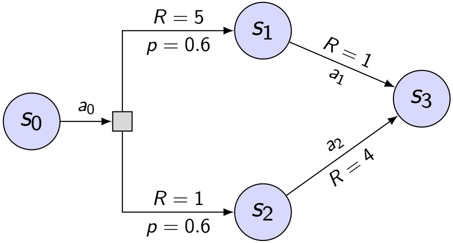
MDP example - blackjack

Blackjack

- S: Possible player hands and played cards
- A: Hit, Stand, ...
- T: Possible drawn cards,
- R: Win/loose at the end

Example: Abstract example

- S: S_0, S_1, S_2, S_3
- A: a_0, a_1, a_2
- T:
$$T(S_0, a_0, S_1) = 0.6$$
$$T(S_0, a_0, S_2) = 0.4$$
$$T(S_1, a_1, S_3) = 1$$
$$T(S_2, a_2, S_3) = 1$$
- R:
$$R(S_0, a_0, S_1) = 5$$
$$R(S_0, a_0, S_2) = 2$$
$$R(S_1, a_1, S_3) = 1$$
$$R(S_2, a_2, S_3) = 4$$



[1]Example: [Mausam, Kobolov: Planning With Markov Decision Processes], tikz by K. Kučerová

When MDP might be a good model?

- *Domain with uncertainty* - uncertain utoucomes of actions
- *Sequential decision making* - for *sequences* of decisions
- *Fair Nature* - no one is actively playing against us
- *Full observability, perfect sensors* - we know where agent is
- *Cyclic domain structures* - when states can be revisited

Policy

Def: Policy

Assignment of action to state, $\pi : S \rightarrow A$

- *Partial policy* - e.g. output of robust replanning
- *Complete policy* - domain of π is whole state space S.
- *Stationary policy* - independent of timestep (e.g. emil)
- *Markovian policy* - dependent only on last state

⚠In general, policy can be history dependent and stochastic!

Value function (of a policy)

Def: Value function

Assignment of value to state, $V : S \rightarrow < -\infty, \infty >$

Value function (of a policy)

Def: Value function

Assignment of value to state, $V : S \rightarrow < -\infty, \infty >$

Def: Value function of a policy

Assignment of value to state based on utility of rewards obtained by following policy π from a state, $V^{\pi} : S \rightarrow < -\infty, \infty >$, $V^{\pi}(s) = u(R_1^{\pi_s}, R_2^{\pi_s}, \dots)$

Value function (of a policy)

Def: Value function

Assignment of value to state, $V : S \rightarrow < -\infty, \infty >$

Def: Value function of a policy

Assignment of value to state based on utility of rewards obtained by following policy π from a state, $V^{\pi} : S \rightarrow < -\infty, \infty >$, $V^{\pi}(s) = u(R_1^{\pi_s}, R_2^{\pi_s}, \ldots)$

Def: Optimal MDP solution

Optimal MDP solution is a policy π^* such that value function V^{π^*} called optimal value function dominates all other value functions in all states, $\forall s V^{\pi^*}(s) \geq V^{\pi}(s)$.

Value function (of a policy)

Def: Value function of a policy

Assignment of value to state based on utility of rewards obtained by following policy π from a state, $V^{\pi} : S \rightarrow < -\infty, \infty >$, $V^{\pi}(s) = u(R_1^{\pi_s}, R_2^{\pi_s}, \ldots)$

Def: Optimal MDP solution

Optimal MDP solution is a policy π^* such that value function V^{π^*} called optimal value function dominates all other value functions in all states, $\forall s V^{\pi^*}(s) \geq V^{\pi}(s)$.

Questions:

- How can we pick u? Can we choose $u(R_1, R_2, \ldots) = \sum_i R_i$?

Expected linear aditive utility

Def: Expected linear aditive utility

Function $u(R_t, R_{t+1}, \ldots) = \mathbb{E}\left[\sum_{t'=t}^{|D|} \gamma^{t'} R_{t'}\right]$ is expected linear additive utility

Sounds convuluted, but it gives

Bellman equation

$V^\pi(s) = [\sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma V^\pi(s')]]$

Expected linear aditive utility

Def: Expected linear aditive utility

Function $u(R_t, R_{t+1}, \ldots) = \mathbb{E}\left[\sum_{t'=t}^{|D|} \gamma^{t'} R_{t'}\right]$ is expected linear additive utility

Sounds convuluted, but it gives

Bellman equation

$V^\pi(s) = [\sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma V^\pi(s')]]$

- $\gamma \in (0, 1]$ is a discount factor, makes agent prefer earlier rewards.

Expected linear aditive utility

Def: Expected linear aditive utility

Function $u(R_t, R_{t+1}, \ldots) = \mathbb{E}\left[\sum_{t'=t}^{|D|} \gamma^{t'} R_{t'}\right]$ is expected linear additive utility

Sounds convuluted, but it gives

Bellman equation

$V^{\pi}(s) = [\sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma V^{\pi}(s')]]$

- $\gamma \in (0, 1]$ is a discount factor, makes agent prefer earlier rewards.
- Risk-neutral

Expected linear aditive utility

Def: Expected linear aditive utility

Function $u(R_t, R_{t+1}, \ldots) = \mathbb{E}\left[\sum_{t'=t}^{|D|} \gamma^{t'} R_{t'}\right]$ is expected linear additive utility

Sounds convuluted, but it gives

Bellman equation

$V^\pi(s) = [\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^\pi(s')]]$

- $\gamma \in (0,1]$ is a discount factor, makes agent prefer earlier rewards.
- Risk-neutral
- For infinite D and bounded rewards, $\gamma < 1$ gives convergence (why?)

Expected linear aditive utility

Def: Expected linear aditive utility

Function $u(R_t, R_{t+1}, \ldots) = \mathbb{E}\left[\sum_{t'=t}^{|D|} \gamma^{t'} R_{t'}\right]$ is expected linear additive utility

Sounds convuluted, but it gives

Bellman equation

$V^\pi(s) = [\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^\pi(s')]]$

- $\gamma \in (0, 1]$ is a discount factor, makes agent prefer earlier rewards.
- Risk-neutral
- For infinite D and bounded rewards, $\gamma < 1$ gives convergence (why?)
- Implies existence of optimal solution

Example

Bellman equation

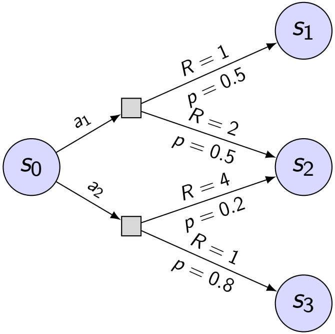
$V^{\pi}(s) = [\sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma V^{\pi}(s')]]$

Example

Bellman equation

$$V^{\pi}(s) = [\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^{\pi}(s')]]$$

Look at the following small MDP. Which action would you take?

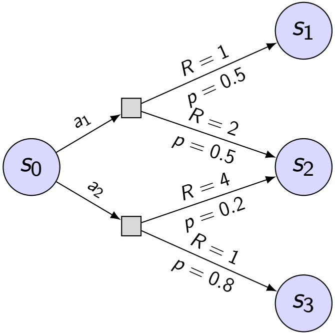


Example

Bellman equation

$V^{\pi}(s) = [\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^{\pi}(s')]]$

Calculate value of a policy $\pi(S_1) = a_1$

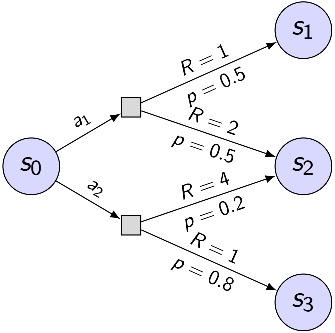


Example

Bellman equation

$V^{\pi}(s) = [\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^{\pi}(s')]]$

Calculate value of a policy $\pi(S_1) = a_2$



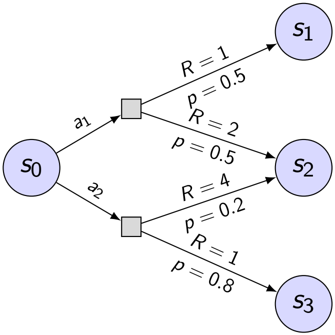
[1] tikz by K. Kučerová (2021)

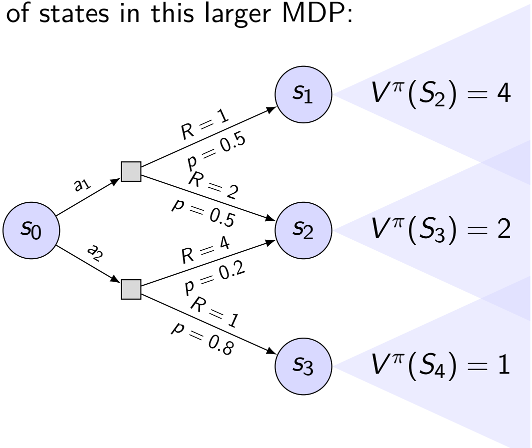
Example

Bellman equation

$V^{\pi}(s) = [\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^{\pi}(s')]]$

Calculate value of both policies given the value of states in this larger MDP:





Optimality principle

When using expected linear additive utility, "MDP" has an optimal deterministic Markovian policy π^*.

Thm: The optimality principle for infinite-horizon MDPs

Infinite horizon MDP with $V^\pi(s_t) = \mathbb{E}\left[\sum_{t'=0}^{\infty} \gamma^{t'} R^\pi_{t+t'}\right]$ and $\gamma \in [0,1)$. Then there exists optimal value function V^*, is stationary, Markovian, and satisfies for all s:

$$V^*(s) = \max_\pi V^\pi(s)$$

$$V^*(s) = \max_{a \in A} \left[\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]\right]$$

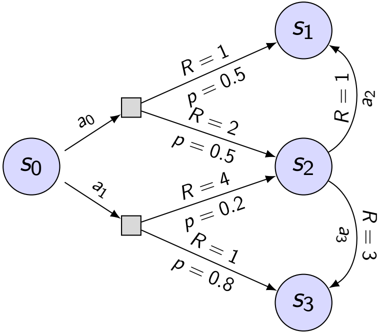
$$\pi^*(s) = \arg\max_{a \in A} \left[\sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]\right]$$

Examples

In the examples, we will use $\gamma = 1$ since we are in domains with finite horizon (and have guaranteed convergence).

Calculate the *optimal* value function in acyclic MDP

- S: S_0, S_1, S_2, S_3
- A: a_0, a_1, a_2, a_3
- T :
$$T(S_0, a_0, S_1) = 0.5$$
$$T(S_0, a_0, S_2) = 0.5$$
$$T(S_1, a_1, S_2) = 0.2$$
$$T(S_2, a_1, S_3) = 0.8$$
$$T(S_2, a_2, S_1) = 1$$
$$T(S_2, a_3, S_3) = 1$$
- R :
$$R(S_0, a_1, S_1) = 1$$
$$R(S_0, a_1, S_2) = 2$$
$$R(S_0, a_2, S_2) = 4$$
$$R(S_0, a_2, S_3) = 1$$
$$R(S_2, a_2, S_1) = 1$$



Calculate the optimal value function in *cyclic* MDP

- S: S_0, S_1, S_2, S_3
- A: a_0, a_1, a_2

  $T(S_0, a_0, S_1) = 0.6$

  $T(S_0, a_0, S_2) = 0.4$
- T : $T(S_1, a_1, S_3) = 1$

  $T(S_2, a_2, S_3) = 0.7$

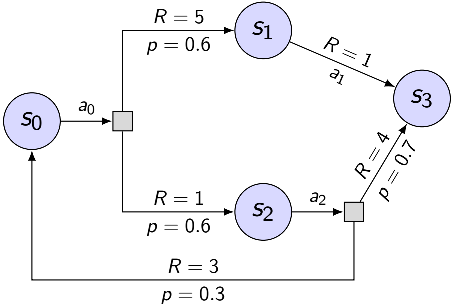
  $T(S_2, a_2, S_0) = 0.3$

  $R(S_0, a_0, S_1) = 5$

  $R(S_0, a_0, S_2) = 2$
- R : $R(S_1, a_1, S_3) = 1$

  $R(S_2, a_2, S_3) = 4$

  $R(S_2, a_2, S_0) = 3$



Calculation notes

Looking at the calculations, what can you say about the calculations of value of function?

Calculation notes

Looking at the calculations, what can you say about the calculations of value of function?

- In acyclic MDP, it's straightforward to calculate the value of states by taking the states in an appropriate order (which is?).

Calculation notes

Looking at the calculations, what can you say about the calculations of value of function?

- In acyclic MDP, it's straightforward to calculate the value of states by taking the states in an appropriate order (which is?).
- Writing the Bellman equations for all states gives a set of linear equations. These can be solved using standard techniques from linear algebra (e.g. substitution :-), do you know other methods or solvers?)

Thank you for participating in the tutorials :-)

Please fill the feedback form →

https://forms.gle/knbzJzHY6MPunLj37