N 1

$$R \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad R R^T = I, \quad \det R = 1$$

$$\Updownarrow$$

$$\begin{bmatrix} r_{11} + r_{13} & r_{12} \\ r_{21} + r_{23} & r_{22} \\ r_{31} + r_{33} & r_{23} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad R R^T = I, \ \det R = 1$$

Let $r_{11} = a$, $r_{21} = b$, $r_{31} = c$. Then

$$R = \begin{bmatrix} a & 1 & -a \\ b & 0 & -b-1 \\ c & 0 & -c+1 \end{bmatrix}.$$

Considering the equations $R R^T = I$ we obtain

$$\begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ -a & -b-1 & -c+1 \end{bmatrix} \begin{bmatrix} a & 1 & -a \\ b & 0 & -b-1 \\ c & 0 & -c+1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$\Downarrow$$

$a = 0$

$$\begin{bmatrix} a & 1 & -a \\ b & 0 & -b-1 \\ c & 0 & -c+1 \end{bmatrix}\begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ -a & -b-1 & -c+1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$\Downarrow$$

$2a^2 + 1 = 1 \to a = 0 \quad (1)$

$b^2 + (-b-1)^2 = 2b^2 + 2b + 1 = 1 \to b = 0 \quad \text{or} \quad b = -1 \quad (2)$

$c^2 + (-c+1)^2 = 2c^2 - 2c + 1 = 1 \to c = 0 \quad \text{or} \quad c = 1 \quad (3)$

$bc + (-b-1)(-c+1) = 2bc - b + c - 1 = 0 \quad (4)$

Using (2), (3), (4) we conclude that only ~~these~~ 2 cases can ~~be~~ occur:

$$\begin{cases} b = 0 \\ c = 1 \end{cases} \quad \text{or} \quad \begin{cases} b = -1 \\ c = 0 \end{cases}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$R_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Looking at the determinants of R_1 and R_2 we see that

$\det R_1 = -1$; $\det R_2 = 1$.

Thus, the only solution ~~to~~ is R_2.

N 2

a) $(R-I)^2\vec{x}_\beta = -(R-I)\vec{o}\,'_\beta$

b) $R - I = \frac{1}{3}\begin{bmatrix} -1 & -2 & 1 \\ 1 & -1 & 2 \\ -2 & -1 & -1 \end{bmatrix}$

$$(R-I)^2 = \frac{1}{9}\begin{bmatrix} -1 & -2 & 1 \\ 1 & -1 & 2 \\ -2 & -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \\ 1 & -1 & 2 \\ -2 & -1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 1 & -2 \\ -2 & -1 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 1 & -2 \\ -2 & -1 & -1 \\ 1 & 2 & -1 \end{bmatrix}}_{A}\vec{x}_\beta = \underbrace{\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}}_{\vec{b}}$$

The nullspace of A is generated by a vector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. A particular solution to $A\vec{x}_\beta = \vec{b}$ is $\vec{x}_p = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Thus, all the points on the axis of motion can be described ~~by~~ as

$$\vec{x}_\beta = \langle \vec{v} \rangle + \vec{x}_p = \begin{bmatrix} t \\ -t \\ -t \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

N 3

Using the Rodriguez formula we get

$$R_2 = I + \sin\theta_2 \, [\vec{v}_2]_\times + (1-\cos\theta_2)\,[\vec{v}_2]_\times^2$$

$$[\vec{v}_2]_\times = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad [\vec{v}_2]_\times^2 = \frac{1}{3}\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\sin\theta_2 = \frac{\sqrt{3}}{2}, \quad \cos\theta_2 = \frac{1}{2}$$

$$R_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} =$$

$$= \frac{1}{6}\left(\begin{bmatrix} 6 & & \\ & 6 & \\ & & 6 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right) =$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} = R_1^T.$$

Then the composite rotation equals

$$R = R_2 R_1 = R_1^T R_1 = I.$$

~~The~~ A unit quaternion ~~a~~ which represents it is for example $\vec{q} = [1 \;\; 0 \;\; 0 \;\; 0]^T$.

N 4

$f_1 = xy + yz + 1$

$f_2 = xyz - 1$ $\qquad F = (f_1, f_2, f_3)$

$f_3 = xz + 1$

$$S(f_1, f_2) = \frac{LCM(LT(f_1), LT(f_2))}{LT(f_1)} \cdot f_1 - \frac{LCM(LT(f_1), LT(f_2))}{LT(f_2)} \cdot f_2$$

$$= \frac{xyz}{xy} \cdot (xy + yz + 1) - \frac{xyz}{xyz} \cdot (xyz - 1) =$$

$$= \cancel{xy}z + yz^2 + z - \cancel{xy}z + 1 = yz^2 + z + 1$$

$$S(f_1, f_3) = \frac{xyz}{xy} \cdot (xy + yz + 1) - \frac{xyz}{xz} \cdot (xz + 1) =$$

$$= \cancel{xy}z + yz^2 + z - \cancel{xy}z - y = yz^2 - y + z$$

$$S(f_2, f_3) = \frac{xyz}{xyz} \cdot (xyz - 1) - \frac{xyz}{xz} \cdot (xz + 1) =$$

$$= \cancel{xy}z - 1 - \cancel{xy}z - y = -y - 1$$

$$\overline{S(f_1, f_2)}^F = yz^2 + z + 1 = f_4$$

$$\overline{S(f_1, f_3)}^F = yz^2 - y + z = f_5$$

$$\overline{S(f_2, f_3)}^F = -y - 1 = f_6$$

Since all 3 remainders are non-zero, we add them to F, i.e.

$$F = (f_1, f_2, f_3, f_4, f_5, f_6).$$

We can also notice that

$$f_2 = y \cdot f_3 + f_6,$$
$$f_5 = f_4 + f_6,$$

which means that F is equivalent to the new system (f_1, f_3, f_4, f_6) and we set

$$F = (f_1, f_3, f_4, f_6).$$

$S(f_1, f_3) = yz^2 - y + z$; $\overline{S(f_1, f_3)}^F = 0$

$S(f_1, f_4) = -xz - x + yz^3 + z^2$; $\overline{S(f_1, f_4)}^F = -x - z + 1 = f_7$

$S(f_1, f_6) = x - yz - 1$; $\overline{S(f_1, f_6)}^F = x + z - 1 = -f_7$

$S(f_3, f_4) = -xz - x + yz$; $\overline{S(f_3, f_4)}^F = -x - z + 1 = f_7$

$S(f_3, f_6) = xz - y$; $\overline{S(f_3, f_6)}^F = 0$

$S(f_4, f_6) = z^2 - z - 1$; $\overline{S(f_3, f_6)}^F = z^2 - z - 1 = f_8$

We add to F non-zero remainders, i.e.

$$F = (f_1, f_3, f_4, f_6, f_7, -f_7, f_8),$$

and notice that

$$f_1 = -y \cdot f_7 - f_6$$

$$f_3 = -z \cdot f_7 - f_8$$

$$f_4 = -z^2 \cdot f_6 - f_8$$

This means that F is equivalent to the system (f_6, f_7, f_8) and we set

$$F = (f_6, f_7, f_8).$$

$$S(f_6, f_7) = -yz + x + y; \quad \overline{S(f_6, f_7)}^F = 0$$

$$S(f_6, f_8) = -yz - y - z^2; \quad \overline{S(f_6, f_8)}^F = 0$$

$$S(f_7, f_8) = -xz - x - z^3 + z^2; \quad \overline{S(f_7, f_8)}^F = 0.$$

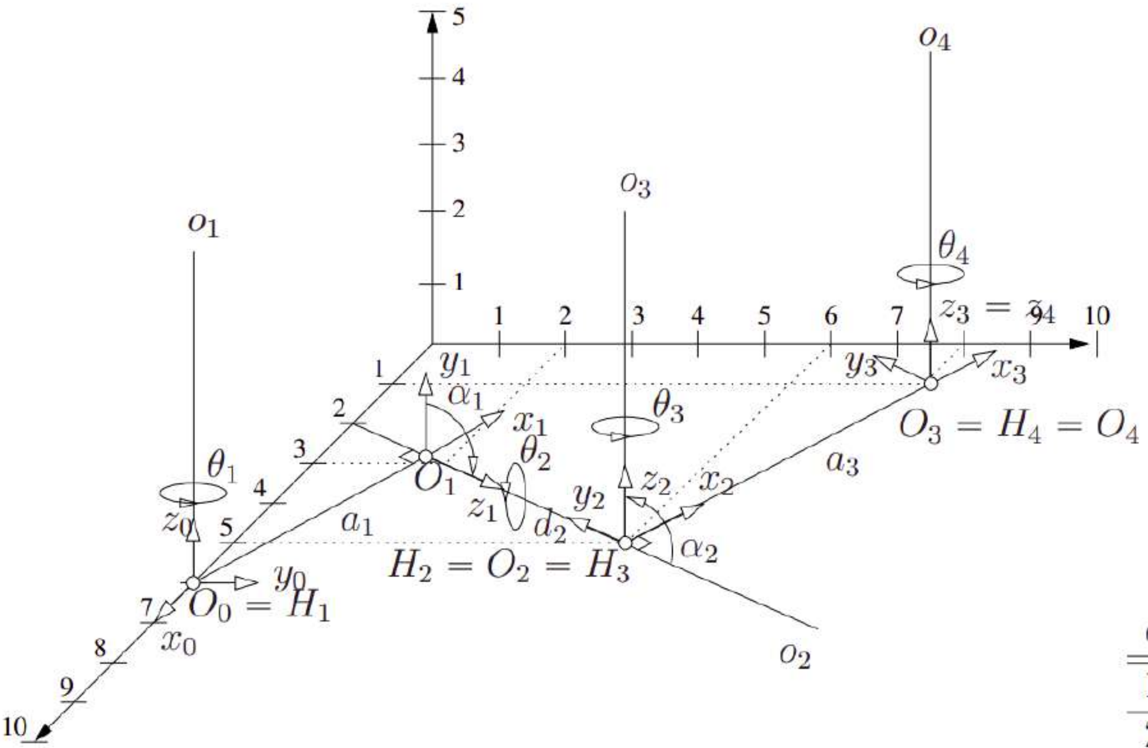
Since all the remainders are zero, we conclude that $F = (f_6, f_7, f_8)$ is a Groebner basis. The solution set S is:

$$f_8 = 0 \longrightarrow z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$f_6 = 0 \longrightarrow y = -1$$

$$f_7 = 0 \longrightarrow x_{1,2} = \frac{1 \mp \sqrt{5}}{2}, \text{ i.e.}$$

$$S = \left\{ \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ -1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} \right\}.$$



o	α	a	θ	d
1	$\frac{\pi}{2}$	a_1	θ_1	0
2	$-\frac{\pi}{2}$	0	θ_2	d_2
3	0	a_3	θ_3	0
4	0	0	θ_4	0