1 Axis of Motion

We will study motion and show that every motion in three dimensional space has an axis of motion. *Axis of motion* is a line of points that remain in the line after the motion. The existence of such an axis will allow us to decompose every motion into a sequence of a rotation around the axis followed by a translation along the axis as shown in Figure 1.1(a).

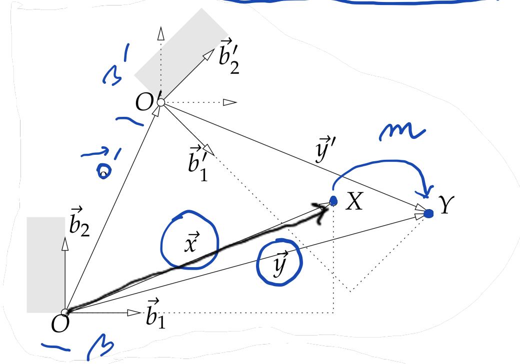
§ 1 Algebraic characterization of the axis of motion. Consider Equation ?? and denote the motion so defined as $m(\vec{x}_\beta) = \mathrm{R}\,\vec{x}_\beta + \vec{o}\,'_\beta$ w.r.t. a fixed coordinate system (O, β). Now let us study the sets of points that remain fixed by the motion, i.e. sets F such that for all $\vec{x}_\beta \in F$ motion m leaves the $m(\vec{x}_\beta)$ in the set, i.e. $m(\vec{x}_\beta) \in F$. Obviously, complete space and the empty set are fixed sets. How do look other, non-trivial, fixed sets?

A nonempty F contains at least one $\vec{x}_\beta$. Then, both $\vec{y}_\beta = m(\vec{x}_\beta)$ and $\vec{z}_\beta = m(\vec{y}_\beta)$ must be in F, see Figure 1.1(b). Let us investigate such fixed points $\vec{x}_\beta$ for which

$$\boxed{\vec{z}_\beta - \vec{y}_\beta = \vec{y}_\beta - \vec{x}_\beta} \tag{1.1}$$

holds true. We do not yet know whether such equality has to necessary hold true for points of all fixed sets F but we see that it holds true for the identity motion *id* that leaves all points unchanged, i.e. $id(\vec{x}_\beta) = \vec{x}_\beta$. We will find later that it holds true for all motions and all their fixed sets.

§ 2 Alibi representation of motion.

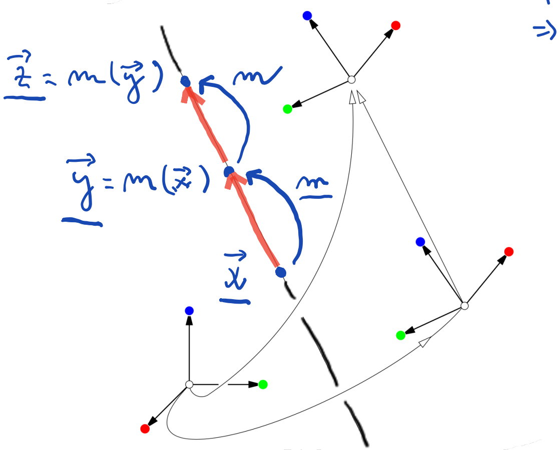


$$
\begin{aligned}
\vec{y}\,'_{\beta'} &= \vec{x}_\beta \\
\vec{y}_{\beta'} - \vec{o}\,'_{\beta'} &= \vec{x}_\beta \\
\mathrm{R}^{-1}\left(\vec{y}_\beta - \vec{o}\,'_\beta\right) &= \vec{x}_\beta \\
\vec{y}_\beta &= \mathrm{R}\,\vec{x}_\beta + \vec{o}\,'_\beta
\end{aligned}
$$

m . mapping

F = fixed set $m(F) = F$

$F \neq \emptyset \Rightarrow \exists \vec{x}_\beta \in F$

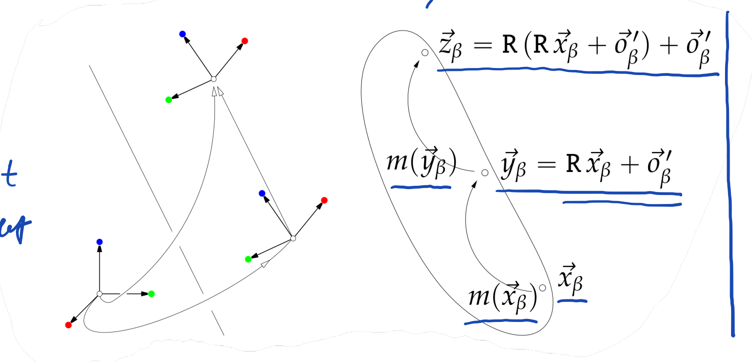


Consider the following sequence of equalities

$$
\begin{aligned}
\vec{z}_\beta - \vec{y}_\beta &= \vec{y}_\beta - \vec{x}_\beta \\
\mathtt{R}\,(\mathtt{R}\,\vec{x}_\beta + \vec{o}'_\beta) + \vec{o}'_\beta - \mathtt{R}\vec{x}_\beta - \vec{o}'_\beta &= \mathtt{R}\,\vec{x}_\beta + \vec{o}'_\beta - \vec{x}_\beta \\
\mathtt{R}^2\vec{x}_\beta + \mathtt{R}\,\vec{o}'_\beta - \mathtt{R}\,\vec{x}_\beta &= \mathtt{R}\,\vec{x}_\beta + \vec{o}'_\beta - \vec{x}_\beta \\
\mathtt{R}^2\vec{x}_\beta - 2\,\mathtt{R}\,\vec{x}_\beta + \vec{x}_\beta &= -\mathtt{R}\,\vec{o}'_\beta + \vec{o}'_\beta \\
\left(\mathtt{R}^2 - 2\,\mathtt{R} + \mathtt{I}\right)\,\vec{x}_\beta &= -(\mathtt{R} - \mathtt{I})\,\vec{o}'_\beta \\
(\mathtt{R} - \mathtt{I})(\mathtt{R} - \mathtt{I})\,\vec{x}_\beta &= -(\mathtt{R} - \mathtt{I})\,\vec{o}'_\beta \qquad (1.2) \\
\boxed{(\mathtt{R} - \mathtt{I})\left((\mathtt{R} - \mathtt{I})\,\vec{x}_\beta + \vec{o}'_\beta\right) = 0} \qquad (1.3)
\end{aligned}
$$

$\vec{x}_\beta \to$ left

$\vec{o}'_\beta \to$ right



Equation 1.3 always has a solution. Let us see why.

Recall that rank $(\mathtt{R} - \mathtt{I})$ is either two or zero. If it is zero, then $\mathtt{R} - \mathtt{I} = 0$ and (i) Equation 1.3 holds for every $\vec{x}_\beta$.

Let rank $(\mathtt{R} - \mathtt{I})$ be two. Vector $\vec{o}'_\beta$ either is zero or it is not zero. If it is zero, then Equation 1.3 becomes $(\mathtt{R} - \mathtt{I})^2\,\vec{x}_\beta = 0$, which has (ii) a one-dimensional space of solutions because the null space and the range of $\mathtt{R} - \mathtt{I}$ intersect only in the zero vector for $\mathtt{R} \neq \mathtt{I}$.

Let $\vec{o}'_\beta$ be non-zero. Vector $\vec{o}'_\beta$ either is in the span of $\mathtt{R} - \mathtt{I}$ or it is not. If $\vec{o}'_\beta$ is in the span of $\mathtt{R} - \mathtt{I}$, then $(\mathtt{R} - \mathtt{I})\,\vec{x}_\beta + \vec{o}'_\beta = 0$ has (iii) one-dimensional affine space of solutions.

If $\vec{o}'_\beta$ is not in the span of $\mathtt{R} - \mathtt{I}$, then $(\mathtt{R} - \mathtt{I})\,\vec{x}_\beta + \vec{o}'_\beta$ for $\vec{x}_\beta \in \mathbb{R}^3$ generates a vector in all one-dimensional subspaces of $\mathbb{R}^3$ which are not in the span of $\mathtt{R} - \mathtt{I}$. Therefore, it generates a non-zero vector $\vec{z}_\beta = (\mathtt{R} - \mathtt{I})\,\vec{y}_\beta + \vec{o}'_\beta$ in the one-dimensional null space of $\mathtt{R} - \mathtt{I}$, because the null space and the span of $(\mathtt{R} - \mathtt{I})$ intersect only in the zero vector for $\mathtt{R} \neq \mathtt{I}$. Equation $(\mathtt{R} - \mathtt{I})\,\vec{z}_\beta = 0$ is satisfied by (iv) a one-dimensional affine set of vectors.

We can conclude that every motion has a fixed line of points for which Equation 1.1 holds. Therefore, every motion has a fixed line of points, every motion has an axis.

1.

2.

rank $(\mathtt{R}-\mathtt{I}) = 0 \Rightarrow \mathtt{R} = \mathtt{I} \Rightarrow 0\,(0\cdot\vec{x}_\beta + \vec{o}'_\beta) = \vec{0} \quad \forall \vec{x}_\beta$

rank $(\mathtt{R}-\mathtt{I}) = 2$

$\vec{o}'_\beta \neq \vec{0}$

$\vec{o}'_\beta \in$ range $(\mathtt{R}-\mathtt{I})$

$\Rightarrow (\mathtt{R}-\mathtt{I})\vec{x}_\beta + \vec{o}'_\beta = 0$

1D affine space

$\Rightarrow (\mathtt{R}-\mathtt{I})((\mathtt{R}-\mathtt{I})\vec{x}_\beta + \vec{o}'_\beta) = 0$

$\vec{o}'_\beta \notin$ range $(\mathtt{R}-\mathtt{I})$

$\vec{o}'_\beta = \vec{0} \Rightarrow (1.3)$:

$(\mathtt{R}-\mathtt{I})\,(\underbrace{(\mathtt{R}-\mathtt{I})}_{\text{rank}=2}\,\underbrace{\vec{x}_\beta}_{\text{1D space}}) = \vec{0}$

$(\mathtt{R}-\mathtt{I})\,\vec{w}_\beta = \vec{0}$

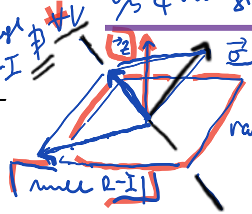
$\vec{w}_\beta \in$ range $(\mathtt{R}-\mathtt{I})$

$\Rightarrow \vec{w}_\beta = \vec{0}$

since

null $(\mathtt{R}-\mathtt{I}) \cap$ range $(\mathtt{R}-\mathtt{I}) = \vec{0}$

$\vec{z} \in$ null $\mathtt{R}-\mathtt{I}$



$\vec{o}' \notin$ span R-I

range R-I $\Rightarrow (\mathtt{R}-\mathtt{I})\,\vec{z}_\beta = \vec{0}$

$\exists \vec{x}: (\mathtt{R}-\mathtt{I})((\mathtt{R}-\mathtt{I})\vec{x}_\beta + \vec{o}'_\beta) = \vec{0}$

§ 2 Geometrical characterization of the axis of motion We now understand the algebraic description of motion. Can we also understand the situation geometrically? Figure 1.2 gives the answer. We shall concentrate on the general situation with $\mathtt{R} \neq \mathtt{I}$ and $\vec{o}_\beta' \neq 0$. The main idea of the figure is that the axis of motion a consists of points that are first rotated away from a by the pure rotation $\mathtt{R}$ around r and then returned back to a by the pure translation $\vec{o}_\beta'$.

$m_R : R, \vec{o}'_\beta = \vec{o}$ vs. $m : R, \vec{o}'_\beta \neq \vec{o}$

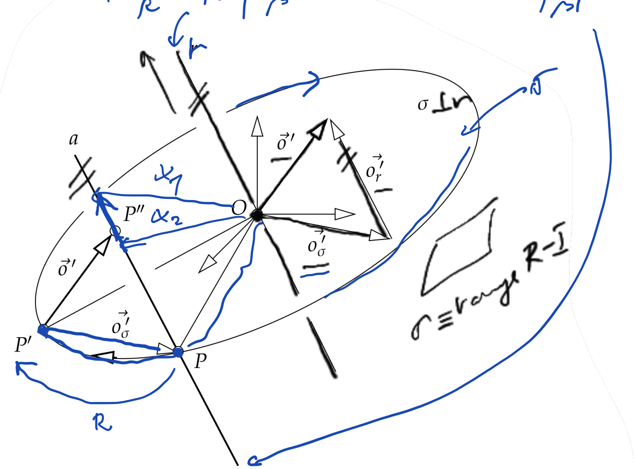


Figure 1.2: Axis a of motion is parallel to the axis of rotation r and intersects the perpendicular plane σ passing through the origin O at a point P, which is first rotated in σ away from a to P' and then returned back to P'' on a by translation $\vec{o}'$. Point P is determined by the component $\vec{o}_\sigma'$ of $\vec{o}'$, which is in the plane σ.

Figure 1.2 shows axis a of motion, which is parallel to the axis of rotation r and intersects the perpendicular plane σ passing through the origin O at a point P, which is first rotated in σ away from a to P' and then returned back to P'' on a by translation $\vec{o}_\beta'$. Point P is determined by the component $\vec{o}_{\sigma\beta}'$ of $\vec{o}_\beta'$, which is in the plane σ. Notice that every vector $\vec{o}_\beta'$ can be written as a sum of its component $\vec{o}_{r\beta}'$ parallel to r and component $\vec{o}_{\sigma\beta}'$ perpendicular to r.

§ 3 Motion axis is parallel to rotation axis. Let us verify algebraically that the rotation axis r is parallel to the motion axis a. Consider Equation 1.2, which we can rewrite as

$x \in a$

$$a \longrightarrow \quad (\mathtt{R}-\mathtt{I})^2\,\vec{x}_\beta = -(\mathtt{R}-\mathtt{I})\,\vec{o}_\beta' \tag{1.4}$$

$(1.2) \quad (R-I)\left((R-I)\vec{x}_\beta + \vec{o}'_\beta\right) = \vec{o}$

Define axis r of motion as the set of points that are left fixed by the pure rotation $\mathtt{R}$, i.e.

$$r \longrightarrow \quad (\mathtt{R}-\mathtt{I})\,\vec{x}_\beta = 0 \tag{1.5}$$

$$\boxed{\mathtt{R}\,\vec{x}_\beta = \vec{x}_\beta} \tag{1.6}$$

$x_1 = 1$

$(\vec{x}_{1\beta} - \vec{x}_{2\beta})$

These are eigenvectors of $\mathtt{R}$ and the zero vector. Take any two solutions $\vec{x}_{1\beta}$, $\vec{x}_{2\beta}$ of Equation 1.4 and evaluate

$$(\mathtt{R}-\mathtt{I})^2(\underbrace{\vec{x}_{1\beta}-\vec{x}_{2\beta}}_{w}) = -(\mathtt{R}-\mathtt{I})\,\vec{o}_\beta' + (\mathtt{R}-\mathtt{I})\,\vec{o}_\beta' = 0 \tag{1.7}$$

since $(R-I)^2\,\vec{w}_\beta = \vec{o}$ (1.7)

$\Rightarrow (R-I)\,\vec{w}_\beta = \vec{o}$

as range $R-I$ $\cap$ null $R-I = \vec{o}$

and thus a non-zero $\vec{x}_{1\beta} - \vec{x}_{2\beta}$ is an eigenvector of $\mathtt{R}$. We see that the direction vectors of a lie in the subspace of direction vectors of r.