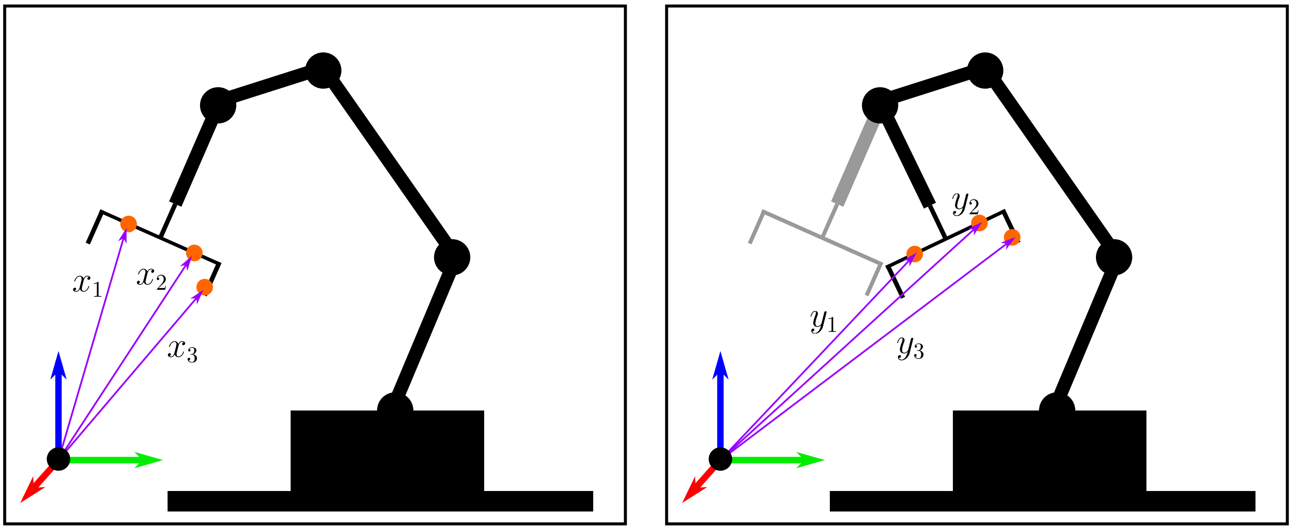
Determining Motion Axes of Robot from 3D Point Measurements

Description



In this task, we study the **axis of motion**. Our task will be to find axes of motion for a 4 DoF robot with 4 revolute joints. This is one of the ways to calibrate a robot, i.e. compute its DH parameters, since from knowing joint axes we may assign coordinate frames to them and computing DH parameters becomes then trivial. In order to compute joint axes, we rigidly attach three markers to the end-effector. Next, we move one joint at a time and measure the starting positions of the markers $\vec{x}_{1\beta}, \vec{x}_{2\beta}, \vec{x}_{3\beta}$ and the positions after the motion $\vec{y}_{1\beta}, \vec{y}_{2\beta}, \vec{y}_{3\beta}$. From these measurements we can estimate rotation $\mathbf{R}$ and translation $\vec{o}\,'_\beta$ of a motion caused by moving one of the joints. Finally, from $\mathbf{R}$ and $\vec{o}\,'_\beta$ we can recover the axis of motion for this joint.

Formulation

Suppose we have a point X attached to the rigid body that moves in $\mathbb{R}^3$. After the motion the point is denoted by Y. Suppose we have an orthonormal coordinate system (O, β) with $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ in which we measure the coordinates of X and Y. Let's denote by (O', β') the new coordinate system that was obtained by moving (O, β) altogether with the moving object (see Figure 1). Notice that the setting is exactly the same as in [1, Figure 5.1 (b)] and [1, Chapter 5.1, §2] (we just attach a point to the rigid body, not the coordinate system). Thus, we may apply [1, Equation (5.5)]:

$$\vec{y}_\beta = \mathbf{R}\vec{x}_\beta + \vec{o}\,'_\beta$$

where $\mathbf{R}$ is a rotation matrix and $\vec{o}\,'_\beta = \overrightarrow{OO'}_\beta$. By measuring coordinates of 3 points attached to the end-effector before and after the motion we obtain 3 equations (in the noiseless case):

$$\vec{y}_{1\beta} = \mathbf{R}\vec{x}_{1\beta} + \vec{o}\,'_\beta, \quad \vec{y}_{2\beta} = \mathbf{R}\vec{x}_{2\beta} + \vec{o}\,'_\beta, \quad \vec{y}_{3\beta} = \mathbf{R}\vec{x}_{3\beta} + \vec{o}\,'_\beta$$

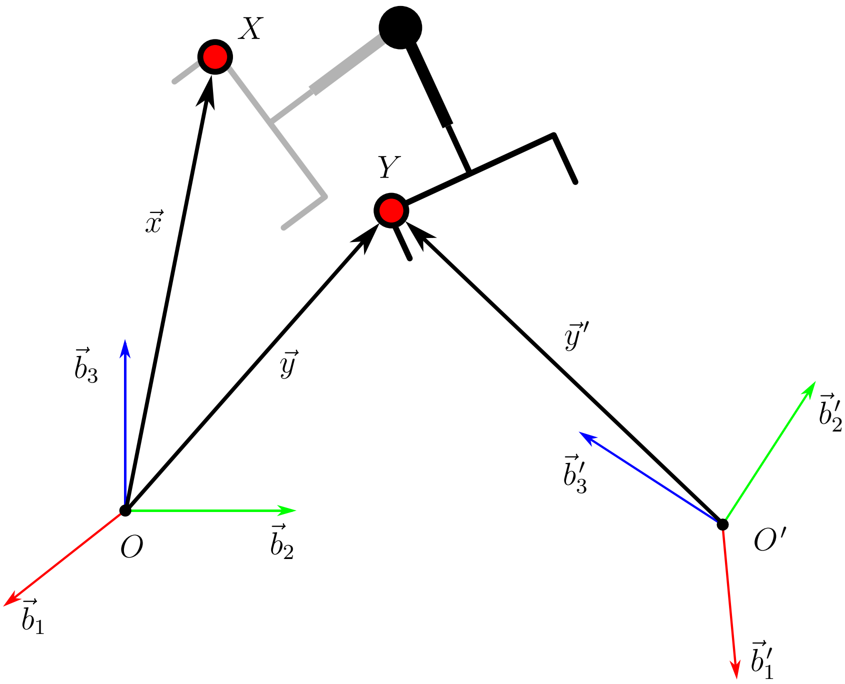


Figure 1: Alibi representation of rigid motion

from which we can determine unknown $\mathbf{R}$ and $\vec{o}\,'_\beta$ and then the axis of motion according to [1, Chapter 9]. In the presence of noise in the measurements $\vec{x}_{i\beta}$ and $\vec{y}_{i\beta}$, we solve the optimization problem:

$$\mathbf{R}^*, \vec{o}\,'^*_\beta = \underset{\substack{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R}) \\ \vec{o}\,'_\beta\in\mathbb{R}^3}}{\arg\min} \underbrace{\sum_{i=1}^{3} \left\| \vec{y}_{i\beta} - \mathbf{R}\vec{x}_{i\beta} - \vec{o}\,'_\beta \right\|_2^2}_{f(\mathbf{R},\vec{o}\,'_\beta)} \tag{1}$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

Solution

According to Appendix, the solution to (1) is given by

$$\mathbf{R}^* = \mathbf{U}\begin{bmatrix} 1 & & \\ & 1 & \\ & & \det \mathbf{U}\mathbf{V}^\top \end{bmatrix}\mathbf{V}^\top, \quad \vec{o}\,'^*_\beta = \widetilde{\mathbf{y}} - \mathbf{R}^*\widetilde{\mathbf{x}},$$

where

$$\widetilde{\mathbf{x}} = \frac{1}{3}\sum_{i=1}^{3}\vec{x}_{i\beta}, \quad \widetilde{\mathbf{y}} = \frac{1}{3}\sum_{i=1}^{3}\vec{y}_{i\beta}$$

and $\mathbf{U}\mathbf{D}\mathbf{V}^\top$ is the singular value decomposition of $\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top$ for

$$\widetilde{\mathbf{X}} = \begin{bmatrix}\vec{x}_{1\beta} & \vec{x}_{2\beta} & \vec{x}_{3\beta}\end{bmatrix} - \widetilde{\mathbf{x}}\mathbf{h}^\top, \quad \widetilde{\mathbf{Y}} = \begin{bmatrix}\vec{y}_{1\beta} & \vec{y}_{2\beta} & \vec{y}_{3\beta}\end{bmatrix} - \widetilde{\mathbf{y}}\mathbf{h}^\top, \quad \mathbf{h} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$$

Next, we can use the material from [1, Chapter 9] to find the axis of motion from $\mathbf{R}^*$ and $\vec{o}\,'^*_\beta$.

References

[1] Tomas Pajdla, *Elements of geometry for robotics*, https://cw.fel.cvut.cz/b221/_media/courses/pkr/pro-lecture-2021.pdf.

Appendix

We will describe the solution to (1) for the general case, namely, when the number of measured points is n. For the sake of simplicity we denote

$$\mathbf{X} = \begin{bmatrix}\vec{x}_{1\beta} & \dots & \vec{x}_{n\beta}\end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix}\vec{y}_{1\beta} & \dots & \vec{y}_{n\beta}\end{bmatrix}, \quad \mathbf{t} = \vec{o}\,'_{\beta}$$

$$\mathbf{h} = \underbrace{\begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^\top}_{n \text{ times}}, \quad \widetilde{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \vec{x}_{i\beta} = \frac{1}{n}\mathbf{X}\mathbf{h}, \quad \widetilde{\mathbf{y}} = \frac{1}{n}\sum_{i=1}^{n} \vec{y}_{i\beta} = \frac{1}{n}\mathbf{Y}\mathbf{h} \tag{2}$$

$$\mathbf{H} = \mathbf{I} - \frac{1}{n}\mathbf{h}\mathbf{h}^\top, \quad \widetilde{\mathbf{X}} = \mathbf{X} - \widetilde{\mathbf{x}}\mathbf{h}^\top = \mathbf{X}\mathbf{H}, \quad \widetilde{\mathbf{Y}} = \mathbf{Y} - \widetilde{\mathbf{y}}\mathbf{h}^\top = \mathbf{Y}\mathbf{H} \tag{3}$$

We first remind that the Frobenius norm of a real matrix is defined by

$$\|\mathbf{A}\|_{\mathrm{F}} \overset{\text{def}}{=} \sqrt{\operatorname{tr}\left(\mathbf{A}^\top\mathbf{A}\right)} \tag{4}$$

and one of its properties is:

$$\|\mathbf{A}+\mathbf{B}\|_{\mathrm{F}}^2 = \|\mathbf{A}\|_{\mathrm{F}}^2 + \|\mathbf{B}\|_{\mathrm{F}}^2 + 2\cdot\operatorname{tr}\left(\mathbf{A}^\top\mathbf{B}\right) \tag{5}$$

Also suppose $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$. Then

$$\operatorname{tr}\left(\mathbf{A}\mathbf{B}\right) = \operatorname{tr}\left(\mathbf{B}\mathbf{A}\right) \tag{6}$$

We now get back to solving (1). We rewrite the criterion function $f(\mathbf{R}, \vec{o}\,'_{\beta})$ in (1) as follows:

$$\begin{aligned}
f(\mathbf{R}, \vec{o}\,'_{\beta}) = \sum_{i=1}^{n} \left\|\vec{y}_{i\beta} - \mathbf{R}\vec{x}_{i\beta} - \vec{o}\,'_{\beta}\right\|_2^2 &\overset{(1)}{=} \sum_{i=1}^{n} \left(\mathbf{y}_i - \mathbf{R}\mathbf{x}_i - \mathbf{t}\right)^\top \left(\mathbf{y}_i - \mathbf{R}\mathbf{x}_i - \mathbf{t}\right) \\
&\overset{(2)}{=} \operatorname{tr}\left(\left(\mathbf{Y} - \mathbf{R}\mathbf{X} - \mathbf{t}\mathbf{h}^\top\right)^\top \left(\mathbf{Y} - \mathbf{R}\mathbf{X} - \mathbf{t}\mathbf{h}^\top\right)\right) \\
&\overset{(3)}{=} \left\|\mathbf{Y} - \mathbf{R}\mathbf{X} - \mathbf{t}\mathbf{h}^\top\right\|_{\mathrm{F}}^2 \\
&\overset{(4)}{=} \left\|\widetilde{\mathbf{Y}} + \widetilde{\mathbf{y}}\mathbf{h}^\top - \mathbf{R}\widetilde{\mathbf{X}} - \mathbf{R}\widetilde{\mathbf{x}}\mathbf{h}^\top - \mathbf{t}\mathbf{h}^\top\right\|_{\mathrm{F}}^2 \\
&= \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}} + \underbrace{\left(\widetilde{\mathbf{y}} - \mathbf{R}\widetilde{\mathbf{x}} - \mathbf{t}\right)}_{\mathbf{t}'}\mathbf{h}^\top\right\|_{\mathrm{F}}^2 \\
&= \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}} + \mathbf{t}'\mathbf{h}^\top\right\|_{\mathrm{F}}^2 \\
&\overset{(5)}{=} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}^2 + \left\|\mathbf{t}'\mathbf{h}^\top\right\|_{\mathrm{F}}^2 + 2\cdot\operatorname{tr}\left(\left(\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right)^\top \mathbf{t}'\mathbf{h}^\top\right) \\
&\overset{(6)}{=} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}^2 + \left\|\mathbf{t}'\mathbf{h}^\top\right\|_{\mathrm{F}}^2 + 2\cdot\operatorname{tr}\left(\mathbf{h}^\top\left(\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right)^\top \mathbf{t}'\right) \\
&\overset{(7)}{=} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}^2 + \left\|\mathbf{t}'\mathbf{h}^\top\right\|_{\mathrm{F}}^2 + 2\cdot\operatorname{tr}\left(\mathbf{h}^\top\mathbf{H}^\top\left(\mathbf{Y} - \mathbf{R}\mathbf{X}\right)^\top \mathbf{t}'\right) \\
&\overset{(8)}{=} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}^2 + \left\|\mathbf{t}'\mathbf{h}^\top\right\|_{\mathrm{F}}^2
\end{aligned}$$

where

(i) $\overset{(1)}{=}$ holds by the definition of Euclidean norm $\|\cdot\|_2$

(ii) $\overset{(2)}{=}$ holds since $\sum_{i=1}^{n} \mathbf{a}_i^\top \mathbf{a}_i = \operatorname{tr}\left(\mathbf{A}^\top \mathbf{A}\right)$, where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$

(iii) $\overset{(3)}{=}$ holds according to (4)

(iv) $\overset{(4)}{=}$ holds according to (3)

(v) $\overset{(5)}{=}$ holds according to (5)

(vi) $\overset{(6)}{=}$ holds according to (6)

(vii) $\overset{(7)}{=}$ holds according to (3)

(viii) $\overset{(8)}{=}$ holds since $\mathbf{h}^\top \mathbf{H}^\top = \mathbf{h}^\top \left(\mathbf{I} - \frac{1}{n}\mathbf{h}\mathbf{h}^\top\right)^\top = \mathbf{h}^\top \left(\mathbf{I} - \frac{1}{n}\mathbf{h}\mathbf{h}^\top\right) = \mathbf{h}^\top - \frac{1}{n} \cdot n\mathbf{h}^\top = \mathbf{0}^\top$ and $\operatorname{tr}\left(\mathbf{0}^\top \mathbf{a}\right) = 0$ for every $\mathbf{a} \in \mathbb{R}^n$.

Now, optimization problem (1) can be reformulated as

$$\mathbf{R}^*, \mathbf{t}'^* = \underset{\substack{\mathbf{R} \in \mathrm{SO}(3,\mathbb{R}) \\ \mathbf{t}' \in \mathbb{R}^3}}{\arg\min} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 + \left\|\mathbf{t}'\mathbf{h}^\top\right\|_\mathrm{F}^2$$

$$\mathbf{t}^* = \widetilde{\mathbf{y}} - \mathbf{R}^*\widetilde{\mathbf{x}} - \mathbf{t}'^*$$

since there are no constraints on $\mathbf{t}'$. Since

$$\min_{\substack{\mathbf{R} \in \mathrm{SO}(3,\mathbb{R}) \\ \mathbf{t}' \in \mathbb{R}^3}} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 + \left\|\mathbf{t}'\mathbf{h}^\top\right\|_\mathrm{F}^2 = \min_{\mathbf{R} \in \mathrm{SO}(3,\mathbb{R})} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 + \min_{\mathbf{t}' \in \mathbb{R}^3} \left\|\mathbf{t}'\mathbf{h}^\top\right\|_\mathrm{F}^2$$

then

$$\mathbf{R}^* = \underset{\mathbf{R} \in \mathrm{SO}(3,\mathbb{R})}{\arg\min} \left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 \tag{7}$$

$$\mathbf{t}'^* = \underset{\mathbf{t}' \in \mathbb{R}^3}{\arg\min} \left\|\mathbf{t}'\mathbf{h}^\top\right\|_\mathrm{F}^2$$

Obviously, $\left\|\mathbf{t}'\mathbf{h}^\top\right\|_\mathrm{F}^2 \geq 0$ attains its minimum for

$$\mathbf{t}'^* = \mathbf{0} \iff \mathbf{t}^* = \widetilde{\mathbf{y}} - \mathbf{R}^*\widetilde{\mathbf{x}}$$

It remains to determine $\mathbf{R}^*$. Problem (7) is a modified well-known orthogonal Procrustes problem (we are just optimizing over specific orthogonal matrices, namely, rotations). We can deduce that

$$\begin{aligned}
\left\|\widetilde{\mathbf{Y}} - \mathbf{R}\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 &\overset{(1)}{=} \left\|\widetilde{\mathbf{Y}}\right\|_\mathrm{F}^2 + \left\|\mathbf{R}\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 - 2 \cdot \operatorname{tr}\left(\widetilde{\mathbf{Y}}^\top \mathbf{R}\widetilde{\mathbf{X}}\right) \\
&\overset{(2)}{=} \left\|\widetilde{\mathbf{Y}}\right\|_\mathrm{F}^2 + \left\|\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 - 2 \cdot \operatorname{tr}\left(\widetilde{\mathbf{Y}}^\top \mathbf{R}\widetilde{\mathbf{X}}\right) \\
&\overset{(3)}{=} \left\|\widetilde{\mathbf{Y}}\right\|_\mathrm{F}^2 + \left\|\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 - 2 \cdot \operatorname{tr}\left(\widetilde{\mathbf{X}}^\top \mathbf{R}^\top \widetilde{\mathbf{Y}}\right) \\
&\overset{(4)}{=} \left\|\widetilde{\mathbf{Y}}\right\|_\mathrm{F}^2 + \left\|\widetilde{\mathbf{X}}\right\|_\mathrm{F}^2 - 2 \cdot \operatorname{tr}\left(\mathbf{R}^\top \widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right)
\end{aligned}$$

where

(i) $\stackrel{(1)}{=}$ holds according to (5)

(ii) $\stackrel{(2)}{=}$ holds since

$$\left\|\mathbf{R}\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}^2 = \mathrm{tr}\left(\left(\mathbf{R}\widetilde{\mathbf{X}}\right)^\top \mathbf{R}\widetilde{\mathbf{X}}\right) = \mathrm{tr}\left(\widetilde{\mathbf{X}}^\top \underbrace{\mathbf{R}^\top\mathbf{R}}_{\mathbf{I}}\widetilde{\mathbf{X}}\right) = \mathrm{tr}\left(\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}\right) = \left\|\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}^2$$

(iii) $\stackrel{(3)}{=}$ holds since $\mathrm{tr}\left(\mathbf{A}\right) = \mathrm{tr}\left(\mathbf{A}^\top\right)$

(iv) $\stackrel{(4)}{=}$ holds according to (6)

Now, the optimization problem (7) is equivalent to

$$\mathbf{R}^* = \underset{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R})}{\arg\max}\ \mathrm{tr}\left(\mathbf{R}^\top\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right) \tag{8}$$

since $\left\|\widetilde{\mathbf{X}}\right\|_{\mathrm{F}}$ and $\left\|\widetilde{\mathbf{Y}}\right\|_{\mathrm{F}}$ don't depend on $\mathbf{R}$.

Remark. *The optimization problem in* (8) *is known in the literature as the problem of finding the closest rotation to a given matrix in the sense of Frobenius norm since* (8) *can we rewritten as*

$$\begin{aligned}
\mathbf{R}^* &= \underset{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R})}{\arg\max}\ \mathrm{tr}\left(\mathbf{R}^\top\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right) \\
&= \underset{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R})}{\arg\min}\ -2\cdot\mathrm{tr}\left(\mathbf{R}^\top\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right) \\
&= \underset{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R})}{\arg\min}\ \mathrm{tr}\left(\mathbf{I}\right) + \left\|\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right\|_{\mathrm{F}}^2 - 2\cdot\mathrm{tr}\left(\mathbf{R}^\top\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right) \\
&= \underset{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R})}{\arg\min}\ \left\|\mathbf{R}\right\|_{\mathrm{F}}^2 + \left\|\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right\|_{\mathrm{F}}^2 - 2\cdot\mathrm{tr}\left(\mathbf{R}^\top\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right) \\
&= \underset{\mathbf{R}\in\mathrm{SO}(3,\mathbb{R})}{\arg\min}\ \left\|\mathbf{R} - \widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right\|_{\mathrm{F}}^2
\end{aligned}$$

Let $\mathbf{U}\mathbf{D}\mathbf{V}^\top$ be the singular value decomposition of $\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top$, where $\mathbf{D}$ is diagonal with entries $d_1 \geq d_2 \geq d_3 \geq 0$. Then

$$\begin{aligned}
\mathrm{tr}\left(\mathbf{R}^\top\widetilde{\mathbf{Y}}\widetilde{\mathbf{X}}^\top\right) &= \mathrm{tr}\left(\mathbf{R}^\top\mathbf{U}\mathbf{D}\mathbf{V}^\top\right) \\
&\stackrel{(1)}{=} \mathrm{tr}\left(\mathbf{V}^\top\mathbf{R}^\top\mathbf{U}\mathbf{D}\right) \\
&= \sum_{i=1}^{3} q_{ii}d_i
\end{aligned}$$

where $\stackrel{(1)}{=}$ follows from (6) and q_{ii} are diagonal entries of $\mathbf{Q} = \mathbf{V}^\top\mathbf{R}^\top\mathbf{U}$. Since $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$ are all orthogonal, then so is $\mathbf{Q}$. There may happen 2 cases:

1. $\det\mathbf{U}\mathbf{V}^\top = 1$, i.e. $\mathbf{Q}$ is a rotation. Then, obviously, since every entry of a real rotation matrix is between -1 and 1, then $\sum_{i=1}^{3} q_{ii}d_i$ is maximized when $q_{11} = q_{22} = q_{33} = 1$ (since all d_i are non-negative), i.e. when

$$\mathbf{Q}^* = \mathbf{I} \iff \mathbf{V}^\top\mathbf{R}^{*\top}\mathbf{U} = \mathbf{I} \iff \mathbf{R}^* = \mathbf{U}\mathbf{V}^\top$$

2. $\det \mathbf{U}\mathbf{V}^\top = -1$, i.e. $\mathbf{Q}$ is a reflection. (Remember that every reflection can be written as $\mathbf{Q} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^\top$ for $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$ and $\mathbf{v}^\top\mathbf{v} = 1$). We can see that we are maximizing

$$\sum_{i=1}^{3} q_{ii} d_i = f(\mathbf{v}) = \sum_{i=1}^{3} (1 - 2v_i^2) \cdot d_i$$

subject to $\mathbf{v}^\top\mathbf{v} - 1 = 0$. We define the Lagrangian function

$$\mathcal{L}(\mathbf{v}, \lambda) = f(\mathbf{v}) + \lambda \cdot (\mathbf{v}^\top\mathbf{v} - 1)$$

Maximizing $f(\mathbf{v})$ subject to $\mathbf{v}^\top\mathbf{v} = 1$ means solving the system of polynomial equations

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \mathbf{0}^\top, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

$$2v_1\lambda - 4v_1 d_1 = 0, \quad 2v_2\lambda - 4v_2 d_2 = 0, \quad 2v_3\lambda - 4v_3 d_3 = 0, \quad v_1^2 + v_2^2 + v_3^2 = 1$$
$$v_1(\lambda - 2d_1) = 0, \quad v_2(\lambda - 2d_2) = 0, \quad v_3(\lambda - 2d_3) = 0, \quad v_1^2 + v_2^2 + v_3^2 = 1$$

There may happen 4 different cases (since $d_1 \geq d_2 \geq d_3$):

(a) $d_1 = d_2 = d_3$. Then all the solutions to the above 4 equations are $\lambda = 2d_1$ and $\mathbf{v}$ is any unit vector. In that case the value of the criterion function for every unit $\mathbf{v}$ is

$$f(\mathbf{v}) = \sum_{i=1}^{3} (1 - 2v_i^2) \cdot d_i = d_1 \cdot \sum_{i=1}^{3} (1 - 2v_i^2) = d_1 \cdot \left(3 - 2 \cdot (v_1^2 + v_2^2 + v_3^2)\right) = d_1$$

Thus, it doesn't matter which $\mathbf{v}$ (or, equivalently, $\mathbf{Q}$) to take. We take, e.g. $\mathbf{v} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, i.e.

$$\mathbf{Q}^* = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \iff \mathbf{R}^* = \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{V}^\top$$

(b) $d_1 = d_2 > d_3$. All the solutions are $\lambda = 2d_1, v_3 = 0, v_1^2 + v_2^2 = 1$ or $v_1 = v_2 = 0, v_3 = 1, \lambda = 2d_3$. In the first case the value of $f(\mathbf{v})$ is

$$f(\mathbf{v}) = \sum_{i=1}^{3} (1 - 2v_i^2) \cdot d_i = d_1 \cdot \left(2 - 2 \cdot (v_1^2 + v_2^2)\right) + d_3 = d_3$$

In the second case the value is

$$f(\mathbf{v}) = \sum_{i=1}^{3} (1 - 2v_i^2) \cdot d_i = d_1 + d_2 - d_3$$

Since in the second case the value $d_1 + d_2 - d_3$ is greater than the value d_3 from the first case, then the solution is $\mathbf{v} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, i.e.

$$\mathbf{Q}^* = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \iff \mathbf{R}^* = \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{V}^\top$$

(c) $d_1 > d_2 = d_3$. All the solutions are $\lambda = 2d_1, v_2 = v_3 = 0, v_1 = 1$ or $\lambda = 2d_2, v_1 = 0, v_2^2 + v_3^2 = 1$. In the first case the value of $f(\mathbf{v})$ is

$$f(\mathbf{v}) = \sum_{i=1}^{3}(1 - 2v_i^2) \cdot d_i = -d_1 + d_2 + d_3$$

In the second case the value is

$$f(\mathbf{v}) = \sum_{i=1}^{3}(1 - 2v_i^2) \cdot d_i = d_1$$

Since in the second case the value d_1 is greater that the value $-d_1 + d_2 + d_3$ from the first case, then $v_1 = 0$ and we take, e.g. $v_2 = 0, v_3 = 1$, i.e.

$$\mathbf{Q}^* = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \iff \mathbf{R}^* = \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{V}^\top$$

(d) $d_1 > d_2 > d_3$. All the solutions to the 4 polynomial equations are $\lambda = 2d_1, v_2 = v_3 = 0, v_1 = 1$, or $= 2d_2, v_1 = v_3 = 0, v_2 = 1$, or $\lambda = 2d_3, v_1 = v_2 = 0, v_3 = 1$. In all the 3 cases the values of $f(\mathbf{v}_i)$ are

$$f(\mathbf{v}_1) = -d_1 + d_2 + d_3$$

$$f(\mathbf{v}_2) = d_1 - d_2 + d_3$$

$$f(\mathbf{v}_3) = d_1 + d_2 - d_3$$

Obviously the value in the last case is the largest one, thus the solution is $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, i.e.

$$\mathbf{Q}^* = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \iff \mathbf{R}^* = \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{V}^\top$$

We can see that in all the cases (a) – (d) we obtain the same answer

$$\mathbf{R}^* = \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{V}^\top$$

Answers from both cases 1. and 2. can be written as

$$\mathbf{R}^* = \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \det \mathbf{U}\mathbf{V}^\top \end{bmatrix} \mathbf{V}^\top$$