Description Logics

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Outline

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- How to check they are designed correctly? How to reason about the knowledge inside?
- We need a **formal language**.

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- propositional logic

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- ... what is the meaning of these formulas ?

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- Syntax – to *represent* concepts (*defining symbols*)

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- Proof Theory – to enforce the semantics

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proof theory ($\vdash$) – resolution, tableau

complexity – NP-Complete (Cook theorem)

First Order Predicate Logic

Example

What is the meaning of this sentence ?

$$(\forall x_1)((Student(x_1) \wedge (\exists x_2)(GraduateCourse(x_2) \wedge isEnrolledTo(x_1, x_2))) \Rightarrow (\forall x_3)(isEnrolledTo(x_1, x_3) \Rightarrow GraduateCourse(x_3)))$$

$$Student \sqcap \exists isEnrolledTo.GraduateCourse \sqsubseteq \forall isEnrolledTo.GraduateCourse$$

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complexity – undecidable (Goedel)

Open World Assumption

OWA

FOPL accepts Open World Assumption, i.e. whatever is not known is not necessarily false.

As a result, FOPL is *monotonic*, i.e.

monotonicity

No conclusion can be invalidated by adding extra knowledge.

This is in contrary to relational databases, or Prolog that accept Closed World Assumption.

Towards Description Logics

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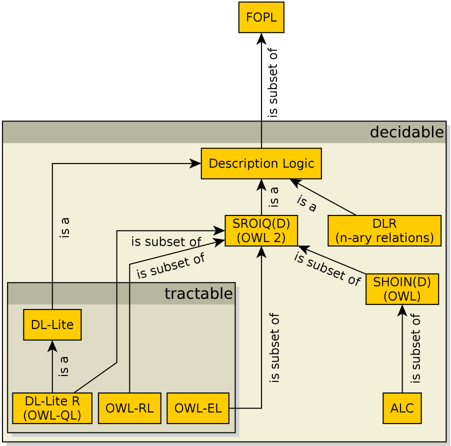
  - We often do not need full expressiveness of FOL.
- Well, we have Prolog – wide-spread and optimized implementation of FOPL, right ?
  - ☹ Prolog is not an implementation of FOPL – OWA vs. CWA, negation as failure, problems in expressing disjunctive knowledge, etc.

What are Description Logics ?



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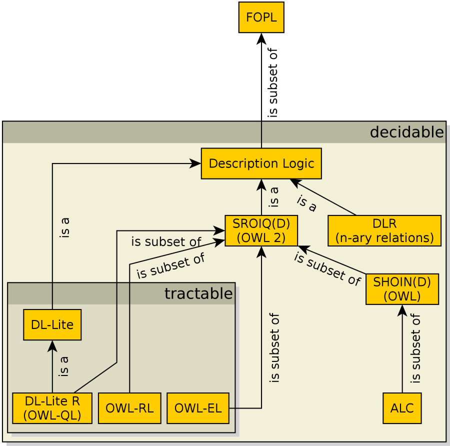
Description logics (DLs) are (almost exclusively) decidable subsets of FOPL aimed at modeling *terminological incomplete knowledge*.



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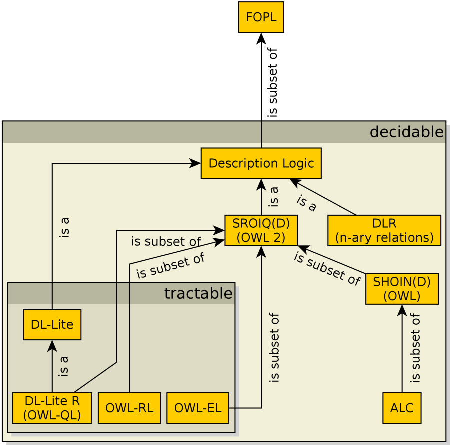

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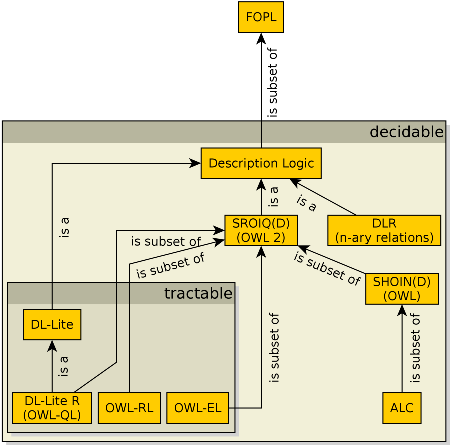

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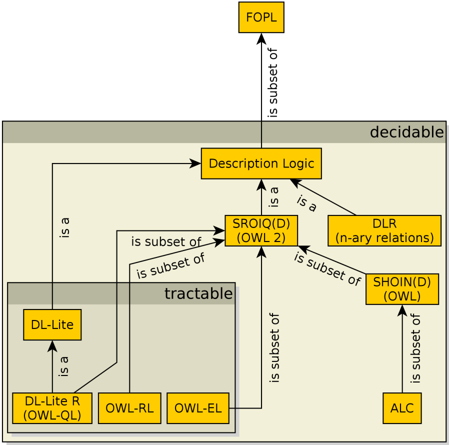

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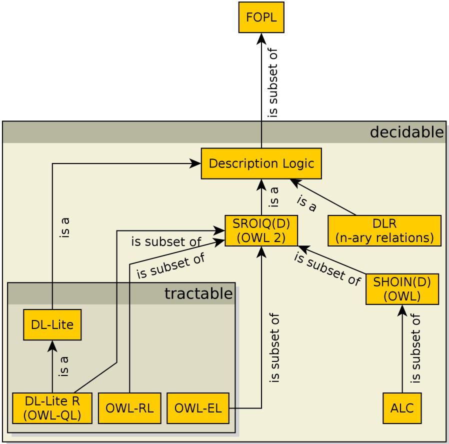

$\mathcal{ALC}$ Language

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- DLs differ in their expressive power (concept/role constructors, axiom types).

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- Having *atomic* concept A, *atomic* role R and individual a, then

$$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$$
$$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$$
$$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$$

$\mathcal{ALC}$ (= attributive language with complements)

Having concepts C, D, atomic concept A and atomic role R, then for interpretation $\mathcal{I}$:

concept	*concept*$^{\mathcal{I}}$	*description*
$\top$	$\Delta^{\mathcal{I}}$	(universal concept)
$\bot$	$\emptyset$	(unsatisfiable concept)
$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$	(negation)
$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$	(intersection)
$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$	(union)
$\forall R \cdot C$	$\{a \mid \forall b((a,b) \in R^{\mathcal{I}} \implies b \in C^{\mathcal{I}})\}$	(universal restriction)
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axiom	$\mathcal{I} \models$ axiom iff	*description*
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$C_1 \equiv C_2$	$C_1^{\mathcal{I}} = C_2^{\mathcal{I}}$	(equivalence)

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ABOX (UNA = unique name assumption[1])

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$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$	(concept assertion)
$R(a_1, a_2)$	$(a_1^{\mathcal{I}}, a_2^{\mathcal{I}}) \in R^{\mathcal{I}}$	(role assertion)

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$\mathcal{ALC}$ – Example

Example

Consider an information system for genealogical data integrating multiple geneological databases. Let's have atomic concepts *Person*, *Man*, *GrandParent* and atomic role *hasChild*.

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- How does the previous axiom look like in FOPL ?

$$\forall x \, (GrandParent(x) \equiv (Person(x) \wedge \exists y \, (hasChild(x, y) \\ \wedge \exists z \, (hasChild(y, z)))))$$

$\mathcal{ALC}$ Example – $\mathcal{T}$

Example

$$
\begin{aligned}
Woman &\equiv Person \sqcap Female \\
Man &\equiv Person \sqcap \neg Woman \\
Mother &\equiv Woman \sqcap \exists hasChild \cdot Person \\
Father &\equiv Man \sqcap \exists hasChild \cdot Person \\
Parent &\equiv Father \sqcup Mother \\
Grandmother &\equiv Mother \sqcap \exists hasChild \cdot Parent \\
MotherWithoutDaughter &\equiv Mother \sqcap \forall hasChild \cdot \neg Woman \\
Wife &\equiv Woman \sqcap \exists hasHusband \cdot Man
\end{aligned}
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  - $\Delta^{\mathcal{I}_1} = Man^{\mathcal{I}_1} = Person^{\mathcal{I}_1} = \{\text{John}, \text{Phillipe}, \text{Martin}\}$
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Interpretation – Example

Example

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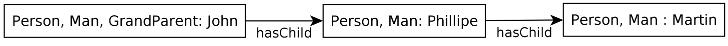
  - $JOHN^{\mathcal{I}_1} = \{\text{John}\}$
- this model is finite and has the form of a tree with the root in the node John :



Shape of DL Models

The last example revealed several important properties of DL models:

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- Both properties represent important characteristics of $\mathcal{ALC}$ that significantly speed-up reasoning.
- In particular (generalized) TMP is a characteristics that is shared by most DLs and significantly reduces their computational complexity.

Example – CWA $\times$ OWA

Example

ABOX

$hasChild(JOCASTA, OEDIPUS)$
$hasChild(OEDIPUS, POLYNEIKES)$
$Patricide(OEDIPUS)$
$hasChild(JOCASTA, POLYNEIKES)$
$hasChild(POLYNEIKES, THERSANDROS)$
$\neg Patricide(THERSANDROS)$

Example – CWA × OWA

Example

ABOX

hasChild(*JOCASTA*, *OEDIPUS*)
hasChild(*OEDIPUS*, *POLYNEIKES*)
Patricide(*OEDIPUS*)

hasChild(*JOCASTA*, *POLYNEIKES*)
hasChild(*POLYNEIKES*, *THERSANDROS*)
¬*Patricide*(*THERSANDROS*)

Edges represent role assertions of *hasChild*; red/green colors distinguish concepts instances – *Patricide* a ¬*Patricide*

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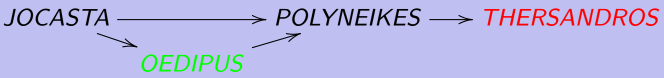

Q1 $(\exists hasChild \cdot (Patricide \sqcap \exists hasChild \cdot \neg Patricide))(JOCASTA)$,

$$JOCASTA \longrightarrow \bullet \longrightarrow \bullet$$

Q2 Find individuals x such that $\mathcal{K} \models C(x)$, where C is

$$\neg Patricide \sqcap \exists hasChild^{-} \cdot (Patricide \sqcap \exists hasChild^{-} \cdot \{JOCASTA\})$$

What is the difference, when considering CWA ?

$$JOCASTA \longrightarrow \bullet \longrightarrow x$$

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