Use an ILP to Solve the Cube of Rubik

March 30, 2020

1 Optimal solution of Rubik's cube with Mixed-Integer Linear Programming

Combinatorial Optimization course, FEE CTU in Prague. Created by Industrial Informatics Department.

In this assigment, we will demonstrate the use of ILP formalism to find the shortest sequence of moves that will solve Rubik's cube from any feasible initial configuration. The inspiration for the model is taken from http://www.m-hikari.com/imf-password2009/45-48-2009/aksopIMF45-48-2009-2.pdf where some mistakes were corrected and the model was a bit improved.

For visualisation and performing moves to the cube, we use nifty package `pycuber` which can be installed simply by

`pip install pycuber`

[1016]:

```
import pycuber as pc
import gurobipy as g
import numpy as np
import itertools as it
```

This is how a solved cube looks like.

[1017]:

```
c = pc.Cube()
c
```

[1017]:

Now, in each turn we can apply 18 different moves. We will randomly scramble our cube by some of the moves:

[1018]:

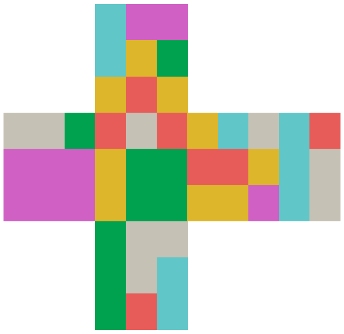
```
moves_mapping = ["B'", "B", "S", "S'", "F", "F'", "U'", "U", "E", "E'", "D",␣
 ↪"D'", "L'", "L", "M'", "M", "R", "R'"]

c = pc.Cube()

# random scramle
alg = ' '.join(np.random.choice(moves_mapping, 5, replace=True))
fc = c.copy()
fc(alg)
print(alg)
c(alg)
```

```
B' L F R F'
```

[1018]:



Now, we will prepare some data for our ILP model. We need to somehow index the individidual subcubes to represent the initial state of the cube.

[1019]:

```
cubes_idxs = np.zeros(shape=(6, 3, 3), dtype=int)

counter = 1
for i in range(3):
    for j in range(3):
        cubes_idxs[0, i, j] = counter
        cubes_idxs[5, i, j] = counter + 45
        counter += 1
for j in range(3):
    for k in range(4):
        cubes_idxs[1+k, 0, j] = 10 + j + k*3
        cubes_idxs[1+k, 1, j] = 10 + j + k*3 + 12
        cubes_idxs[1+k, 2, j] = 10 + j + k*3 + 24

faces = []
for f in range(6):
    faces.append([])
    for i in range(3):
        for j in range(3):
            faces[f] += [cubes_idxs[f, i, j]]
```

```
faces_letters = 'U L F R B D'.split()
colors = ['red', 'yellow', 'green', 'white', 'orange', 'blue']
init_pattern = []
for face_idx, face in enumerate(faces_letters):
    a = c.get_face(face)
    for i in range(3):
        for j in range(3):
            init_pattern += [(cubes_idxs[face_idx, i, j], colors.index(a[i][j].
 ↪colour) + 1)]
```

1.1 ILP model

Finally, we build and solve the model.

[1026]:

```
G = [
    (1,1,18), (1,2,30), (1,3,42), (1,10,3), (1,22,2),
    (1,34,1), (1,18,54), (1,30,53), (1,42,52), (1,52,10), (1,53,22),
    (1,54,34), (1,19,43), (1,20,31), (1,21,19), (1,31,44), (1,33,20),
    (1,43,45), (1,44,33), (1,45,21), (3,4,17), (3,5,29), (3,6,41), (3,11,6),␣
 ↪(3,23,5), (3,35,4), (3,17,51), (3,29,50), (3,41,49),
    (3,49,11), (3,50,23), (3,51,35), (5,7,16), (5,8,28), (5,9,40),
    (5,12,9), (5,24,8), (5,36,7), (5,16,48), (5,28,47), (5,40,46),
    (5,46,12), (5,47,24), (5,48,36), (5,13,15), (5,14,27), (5,15,39),
    (5,25,14), (5,27,38), (5,37,13), (5,38,25), (5,39,37), (7,10,13),
    (7,11,14), (7,12,15), (7,13,16), (7,14,17), (7,15,18), (7,16,19),
    (7,17,20), (7,18,21), (7,19,10), (7,20,11), (7,21,12), (7,3,1),
    (7,6,2), (7,9,3), (7,2,4), (7,8,6), (7,1,7), (7,4,8), (7,7,9),
    (9,22,25), (9,23,26), (9,24,27), (9,25,28), (9,26,29), (9,27,30),
    (9,28,31), (9,29,32), (9,30,33), (9,31,22), (9,32,23), (9,33,24),
    (11,34,37), (11,35,38), (11,36,39), (11,37,40), (11,38,41),
    (11,39,42), (11,40,43), (11,41,44), (11,42,45), (11,43,34),
    (11,44,35), (11,45,36), (11,46,48), (11,47,51), (11,48,54),
    (11,49,47), (11,51,53), (11,52,46), (11,53,49), (11,54,52),
    (13,1,45), (13,4,33), (13,7,21), (13,13,1), (13,25,4), (13,37,7),
    (13,21,52), (13,33,49), (13,45,46), (13,46,13), (13,49,25),
    (13,52,37), (13,10,34), (13,11,22), (13,12,10), (13,22,35),
    (13,24,11), (13,34,36), (13,35,24), (13,36,12), (15,2,44), (15,5,32),
    (15,8,20), (15,14,2), (15,26,5), (15,38,8), (15,20,53), (15,32,50),
    (15,44,47), (15,47,14), (15,50,26), (15,53,38), (17,3,43), (17,6,31),
    (17,9,19), (17,15,3), (17,27,6), (17,39,9), (17,19,54), (17,31,51),
    (17,43,48), (17,48,15), (17,51,27), (17,54,39), (17,16,18),
    (17,17,30), (17,18,42), (17,28,17), (17,30,41), (17,40,16),
    (17,41,28), (17,42,40)
]
```

```
# in theory, with at most 26 moves you can solve the cube from any initial␣
 ↪configuration: http://cube20.org/qtm/
max_moves = 6


max_moves += 1
m = g.Model()
y = m.addVars(18, max_moves, vtype=g.GRB.BINARY, name='y')
x = m.addVars(54, max_moves, vtype=g.GRB.INTEGER, lb=1, ub=6, name='x')
moves_used = m.addVar(vtype=g.GRB.CONTINUOUS, ub=max_moves, obj=1)

# objective: minimize the number of used moves
m.addConstrs(moves_used >= g.quicksum((t+1)*y[i, t] for i in range(18)) for t␣
 ↪in range(max_moves))

for t in range(max_moves-1):
    for k, i, j in G:
        # if we perform move k at time t, then colors of the affected cubes␣
 ↪must change accordingly
        # beware! this is not an ILP constraint. but can be translated to a one␣
 ↪with a big-M
        m.addConstr((y[k-1, t] == 1) >> (x[i-1, t] == x[j-1, t+1]))

        # inverse to the k move
        m.addConstr((y[k+1-1, t] == 1) >> (x[j-1, t] == x[i-1, t+1]))

        # non-affected cubes must remain the same
        m.addConstr(x[i-1, t] - 6*(y[k-1, t] + y[k+1-1, t] +
                                   g.quicksum(y[l-1, t] + y[l+1-1, t] for l, m,␣
 ↪n in G if m==i and k!=l)) <= x[i-1, t+1]
                   )
        m.addConstr(x[i-1, t+1] <= x[i-1, t] + 6*(y[k-1, t] + y[k+1-1, t] +
                                   g.quicksum(y[l-1, t] + y[l+1-1, t] for l, m,␣
 ↪n in G if m==i and k!=l))
                   )

# final state conditions - all cubes in every face must have the same colors
for f in range(6):
    for i in faces[f]:
        for j in faces[f]:
            if i > j:
                m.addConstr(x[i-1, max_moves-1] == x[j-1, max_moves-1])


# set initial cube configuration
for cidx, color in init_pattern:
```

```python
    m.addConstr(x[cidx-1, 0] == color)

# one move at the time
m.addConstrs(g.quicksum(y[i, t] for i in range(18)) <= 1 for t in␣
 ↪range(max_moves))

# redundant constaints:
for t in range(max_moves-1):
    # forbid mirror moves (gives solver extra information) +
    # symmetry breaking: do not use sequence of two moves of the second kind (i.
 ↪e., prime moves)
    # if it has to do it, it will achieve the same effect with two moves of the␣
 ↪first kind
    m.addConstrs(y[2*k+1, t] + y[2*k, t+1] <= 1 for k in range(9))
    m.addConstrs(y[2*k, t] + y[2*k+1, t+1] + y[2*k+1, t] <= 1 for k in range(9))

# solve the problem
m.params.mipfocus = 1  # focus on feasiblity
m.optimize()
```

```
[[1, 2, 3, 4, 5, 6, 7, 8, 9], [10, 11, 12, 22, 23, 24, 34, 35, 36], [13, 14, 15,
25, 26, 27, 37, 38, 39], [16, 17, 18, 28, 29, 30, 40, 41, 42], [19, 20, 21, 31,
32, 33, 43, 44, 45], [46, 47, 48, 49, 50, 51, 52, 53, 54]]
Changed value of parameter mipfocus to 1
   Prev: 0  Min: 0  Max: 3  Default: 0
Gurobi Optimizer version 9.0.0 build v9.0.0rc2 (mac64)
Optimize a model with 2264 rows, 505 columns and 15703 nonzeros
Model fingerprint: 0x8c139283
Model has 1872 general constraints
Variable types: 1 continuous, 504 integer (126 binary)
Coefficient statistics:
  Matrix range     [1e+00, 7e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [1e+00, 7e+00]
  RHS range        [1e+00, 6e+00]
Presolve added 2083 rows and 0 columns
Presolve removed 0 rows and 72 columns
Presolve time: 0.10s
Presolved: 4347 rows, 433 columns, 15294 nonzeros
Variable types: 0 continuous, 433 integer (108 binary)

Root relaxation: objective 6.617673e-01, 2585 iterations, 0.07 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    0.66177    0  310          -    0.66177      -     -    0s
```

```
     0     0    1.00000    0  261          -    1.00000      -     -    0s
     0     0    1.00000    0  257          -    1.00000      -     -    0s
     0     0    2.00000    0  323          -    2.00000      -     -    0s
     0     0    2.00000    0  314          -    2.00000      -     -    0s
     0     0    2.00000    0  339          -    2.00000      -     -    0s
     0     0    2.00000    0  317          -    2.00000      -     -    0s
     0     0    2.00000    0  321          -    2.00000      -     -    0s
     0     0    2.00000    0  342          -    2.00000      -     -    0s
     0     0    2.00000    0  321          -    2.00000      -     -    1s
     0     1    2.00000    0  321          -    2.00000      -     -    4s
   767   539    4.00000   26  187          -    2.00000      -  97.9    5s
  1281   829    4.00000   20  319          -    2.07253      -  99.3   10s
  2312  1079 infeasible   33               -    3.00000      -  73.6   15s
  4172  1379 infeasible   29               -    3.00000      -   104   20s
  5809  2036 infeasible   57               -    3.00000      -   113   25s
  7191  2610 infeasible   40               -    3.00000      -   120   30s
* 7228  2173              48       6.0000000    3.00000  50.0%   121   30s
H 7590  1579                       5.0000000    3.00000  40.0%   122   31s

Cutting planes:
  Cover: 1
  Implied bound: 2
  MIR: 62
  StrongCG: 2
  Flow cover: 75
  Zero half: 3
  RLT: 1
  Relax-and-lift: 2

Explored 9466 nodes (1131289 simplex iterations) in 34.38 seconds
Thread count was 12 (of 12 available processors)

Solution count 2: 5 6

Optimal solution found (tolerance 1.00e-04)
Best objective 5.000000000000e+00, best bound 5.000000000000e+00, gap 0.0000%
```

Let us extract the solution:

[1027]:

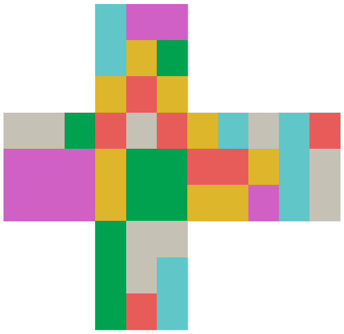
```python
solution = []
for t in range(max_moves):
    for i in range(18):
        if y[i, t].x > 0.5:
            solution += [moves_mapping[i]]
solution_alg = ' '.join(solution)
print(solution_alg)
```

```
F R' F' L' B
```

Ok, so this was our original cube:

[1022]:
```
c
```

[1022]:



And now, we apply the obtain solution algorithm:

[1023]:
```
c(solution)
```

[1023]:

Indeed, the cube is solved! For sake of comparison, let us solve the cube with so-called *Corners-first method* that humans typically use to solve the cube:

[1024]:
```
from pycuber.solver import CFOPSolver

solver = pc.solver.CFOPSolver(fc)
solution = solver.solve(suppress_progress_messages=False)
```

```
Cross: U2 B' R B' D2 R D2
F2L('green', 'orange'): y R U' R' y' U F' U2 F
F2L('orange', 'blue'): y U R U2 R' U R U' R'
F2L('blue', 'red'): y R U' R' U F' U' F
F2L('red', 'green'): y U R U R' U' F' U' F
OLL:  U2 R U R' U R U' R' U' R' F R F'
PLL:  y U x' R U' R' D R U R' D' R U R' D R U' R' D'

FULL: U2 B' R B' D2 R D2 B U' B' U F' U2 F U B U2 B' U B U' B' L U' L' U B' U' B
U F U F' U' L' U' L U2 F U F' U F U' F' U' F' L F L' U R B' R' F R B R' F' R B
R' F R B' R' F'
```

```
[1025]: length_legacy_alg = sum([1 if not '2' in str(a) else 2 for a in solution])
        length_legacy_alg
```

```
[1025]: 73
```

Hence, the suboptimal solution is much longer (but it is much faster to find it).

```
[ ]:
```