Combinatorial Optimization
Lab No. 5
Applications of the Shortest Path Problem

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April 2, 2024

Abstract

In this lab, we present an interesting application of shortest paths and we remind ourselves, that SPT problem is generally $\mathcal{NP}$-hard.

1 Shortest paths with negative cycles

In this example, we demonstrate that Bellman-Ford's algorithm fails to produce correct answer on the following instance, wheres an ILP formulation of SPT yields correct solution.

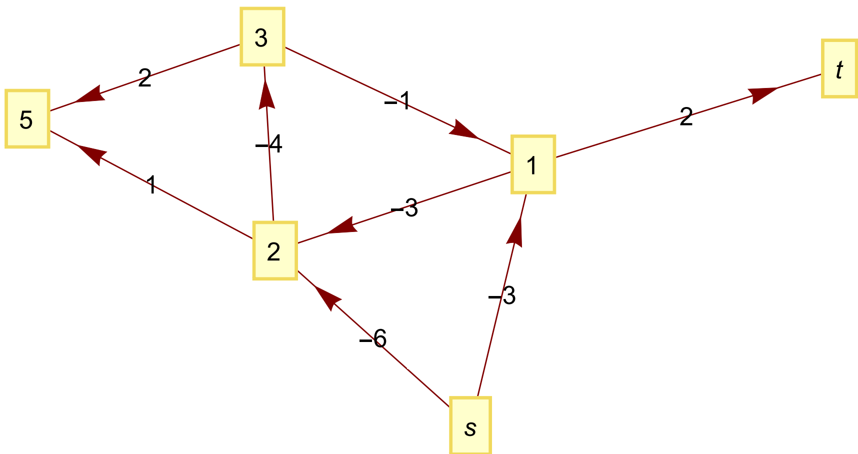

Figure 1: A graph with a negative cycle.

2 Approximation of piecewise-linear functions

Approximation of piecewise linear functions is lesser-known, however quite an interesting application of the shortest path problem [1]. As an input a set of n points is given, i.e. $\{(x_i, f(x_i))\}$.

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x = [   0,   1.26,   2.51,    3.77,    5.03,   6.28]
f = [0.01,   1.16,    0.7,   -0.34,    -0.8,   0.21]
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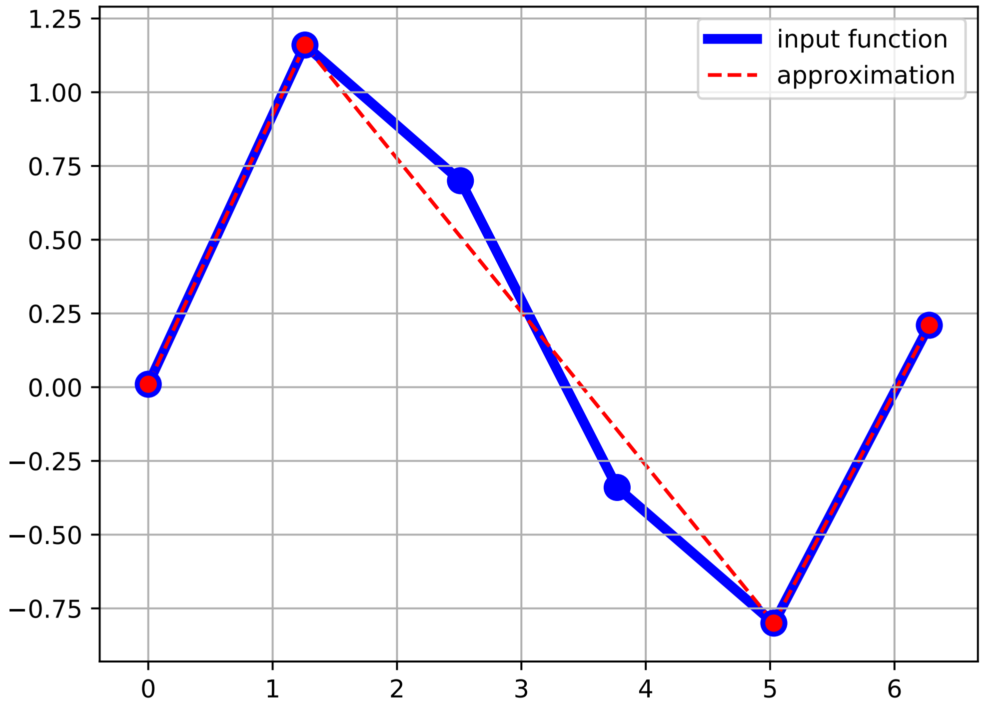

Figure 2: The input function (blue) and the approximated function (red).

The goal is to select the smallest possible number of points on condition that approximation error is reasonable (see Figure 2).

This problem can be solved using the shortest path problem applied to graph G where each node v_i corresponds to a data point $(x_i, f(x_i))$ and each edge $e = (i, j)$ (such that $i < j$) has weight $c_{i,j}$ which corresponds to a penalty if the part of the function from $(x_i, f(x_i))$ to $(x_j, f(x_j))$ is approximated by a line segment. The penalty consists of the approximation error (weighted by β) and the number of required samples (the 1 term). An example of such graph can be seen in Figure 2 and penalty $c_{i,j}$ is calculated as follows.

$$c_{i,j} = 1 + \beta \left[\sum_{k=i}^{j} \left(f(x_k) - f'(x_k, x_i, x_j)\right)^2 \right] \tag{1}$$

$f'(x_k)$ is a precomputed value of the approximating function.

$$f'(x, x_i, x_j) = f(x_i) + (x - x_i) \cdot \frac{f(x_j) - f(x_i)}{x_j - x_i} \tag{2}$$

In a similar way the shortest path problem can be applied to some vector pictures which are composed of points (x_i, y_i). The difference is that weights $c_{i,j}$ are computed as a sum of squared distances from points (x_k, y_k) to the line segment defined by (x_i, y_i) and (x_j, y_j) points where $i < k < j$.

$$c_{i,j} = 1 + \beta \left[\sum_{k=i}^{j} \frac{|(y_j - y_i)x_k - (x_j - x_i)y_k + x_j y_i - y_j x_i|^2}{(y_j - y_i)^2 + (x_j - x_i)^2} \right] \tag{3}$$

An example of two-dimensional approximation, where the frontiers of the Czech Republic are input data, is depicted in Figure 3.

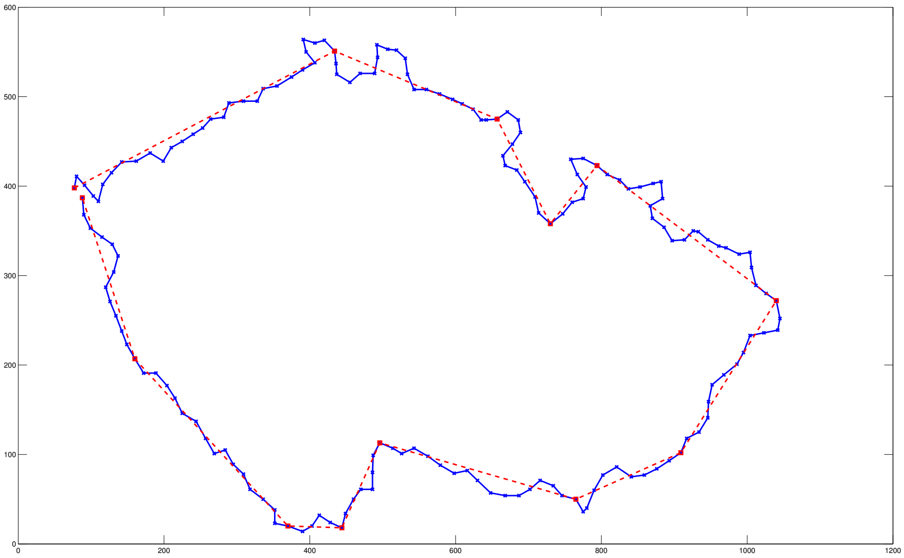

Figure 3: Entered points (blue) and selected points (red).

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications.* Prentice Hall; United States Ed edition, 1993.