Combinatorial Optimization: Lab No. 3
Integer Linear Programming

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March 4, 2024

Abstract

This lab is devoted to formulating problems as ILP, namely Call center scheduling problem. Taking this problem as an example, we show how to formulate problems with absolute value functions in their objectives as ILP.

1 Call center scheduling problem

1.1 Problem Description

A call center needs to create a cyclic daily schedule (i.e., numbers of the shifts at particular hours are same on every day) of the *shifts* for its employees [1]. Let **d** be a vector of a *personnel demand* in each hour during the day, i.e., d_i is defined $\forall i = 0, 1, 2, \ldots, 23$. The personnel demand d_i determines the minimal number of shifts (i.e., employees who will be assigned to these shifts) between i and $(i+1)$ hour, e.g., d_{10} expresses the minimal number of shifts running between 10 a.m. and 11 a.m. The objective of this problem is to obtain the cyclic daily schedule of the shifts such that the total number of the shifts used to cover the personnel demand in a day is minimized. The shift may start at arbitrary full hour and its length is set to 8 hours.

The ILP model of this problem is based on a variable $\mathbf{x}$ such that x_i represents a number of the shifts starting at hour i.

Lab exercise: Formulate this problem as an ILP.

Consider an example with the following vector of the demands

```
d = [6,6,6,6,6,8,9,12,18,22,25,21,21,20,18,21,21,24,24,18,18,18,12,8]
```

Using ILP to solve this example, we find out that the minimal number of the shifts that satisfy this demand is 55 (i.e., optimal value of the objective function). The cyclic daily schedule of the used shifts is illustrated in Fig. 1. Note that there might be several solutions with the same value of the objective function. You can notice the large surplus of the shifts in comparison to the personnel demand between 4 a.m. and 5 a.m. How to minimize this difference between the personnel demand and its coverage is handled in Section 2.

2 A Homework Assignment

To avoid a large surplus of shifts in comparison to the personnel demand, we modify our objective to find a daily schedule of shifts such that the difference of the personnel demand and its coverage

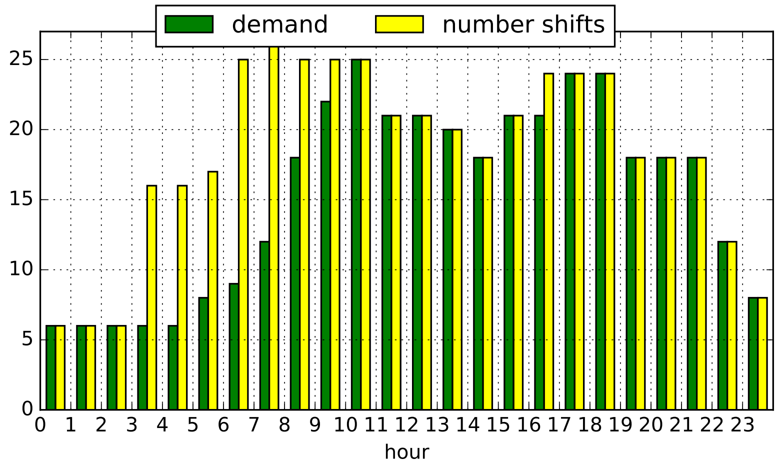

Figure 1: The coverage of the personnel demand $\mathbf{d}$

is minimized, i.e., we try to cover the personnel demand as precisely as possible

$$\min \sum_{i=0}^{23} \left| d_i - \sum_{j=i-7}^{i} x_{j \bmod 24} \right| \tag{1}$$

Unfortunately, the absolute value function in Eq. (1) cannot be used in the ILP model directly. It is necessary to substitute it by an auxiliary variable z that will be bounded by the constraints representing the personnel demand coverage. We can substitute the absolute value by noticing that in general $|x| = \max\{x, -x\} = z$. The maximum function can be removed by imposing additional constraints $x \leq z, -x \leq z$ that ensure that in every optimal solution z is equal to the $\max\{x, -x\}$, and, therefore equal to $|x|$. Please notice that this transformation **does not generally work** if the absolute value was not originally in the objective function.

Since in our case we have a sum of absolute values, variable z_i has to be created for each absolute value. A general transformation is shown in Eq. (3), where $\mathbf{v}, \mathbf{z}$ are vectors of variables and $\mathbf{q}$ is a vector of constants.

$$\begin{array}{llcll}
 & \min \sum_i \left| q_i - v_i \right| & \longrightarrow & \min \sum_i z_i & \longrightarrow & \min \sum_i z_i \\
\text{s.t.} & & & & & \\
 & & \longrightarrow & \begin{array}{rcl} \left| v_i - q_i \right| & = & z_i \\ z_i & \geq & 0 \end{array} & \longrightarrow & \begin{array}{rcl} q_i - v_i & \leq & z_i \\ v_i - q_i & \leq & z_i \\ z_i & \geq & 0 \end{array}
\end{array} \tag{2}$$

Fig. 2 shows the shifts obtained by solving the model Eq. (1) using the data from the example given in Section 1.1. You can see that the number of shifts is closer to the personnel demand coverage.

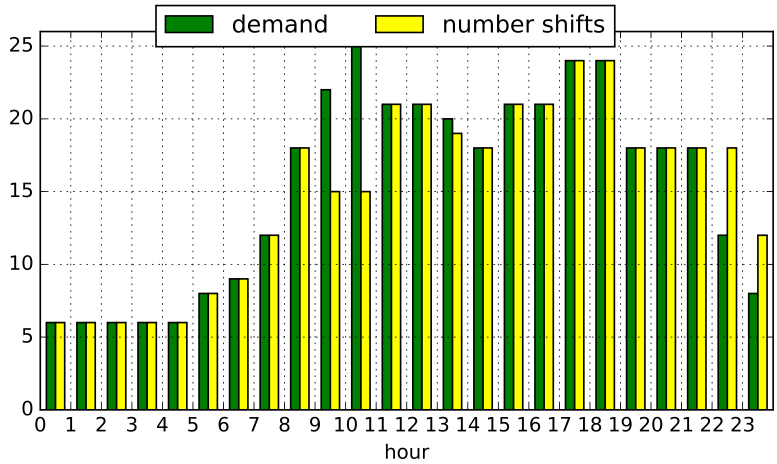

Figure 2: The coverage of the personnel demand $\mathbf{d}$ and the daily schedule minimizing Eq. (1) of the shifts with the value of objective function equal to 28.

However, this modified criterion does not ensure that each hour is sufficiently covered - we could end up with a shift schedule with specific hours that are covered only by a single person. To avoid this, we further constrain the model so the demand does not exceed the number of present staff by more than D.

$$d_i - \sum_{j=i-7}^{i} x_{j \bmod 24} \leq D \quad \forall i \in \{0, \ldots, 23\} \tag{3}$$

Finally, we extend the planning horizon to cover the entire week, including weekends. To achieve this, we increase the number of variables modeling the number of starting jobs at a given hour to $24 \cdot 7 = 168$ to include each hour of the week $x_0, \ldots, x_{167}$. We are given two vectors of length 24 $\mathbf{d}, \mathbf{e}$, which show the demands for weekdays and weekends, respectively. To solve the extended problem, we use demands d_i from vector $\mathbf{d}$ for the first five days of the schedule and demands e_i from vector $\mathbf{e}$ for the remaining two days. Furthermore, the modulo operation modeling the periodic nature of the shifts needs to consider this extended planning horizon as well.

With inputs:

```
d = [6,6,6,6,6,8,9,12,18,22,25,21,21,20,18,21,21,24,24,18,18,18,12,8]

e = [3 3 3 3 3 4 4 6 9 11 12 10 10 10 9 10 10 12 12 9 9 9 6 4]
```

and $D = 2$, the optimal schedule with criterion 232, is shown in Figure 3

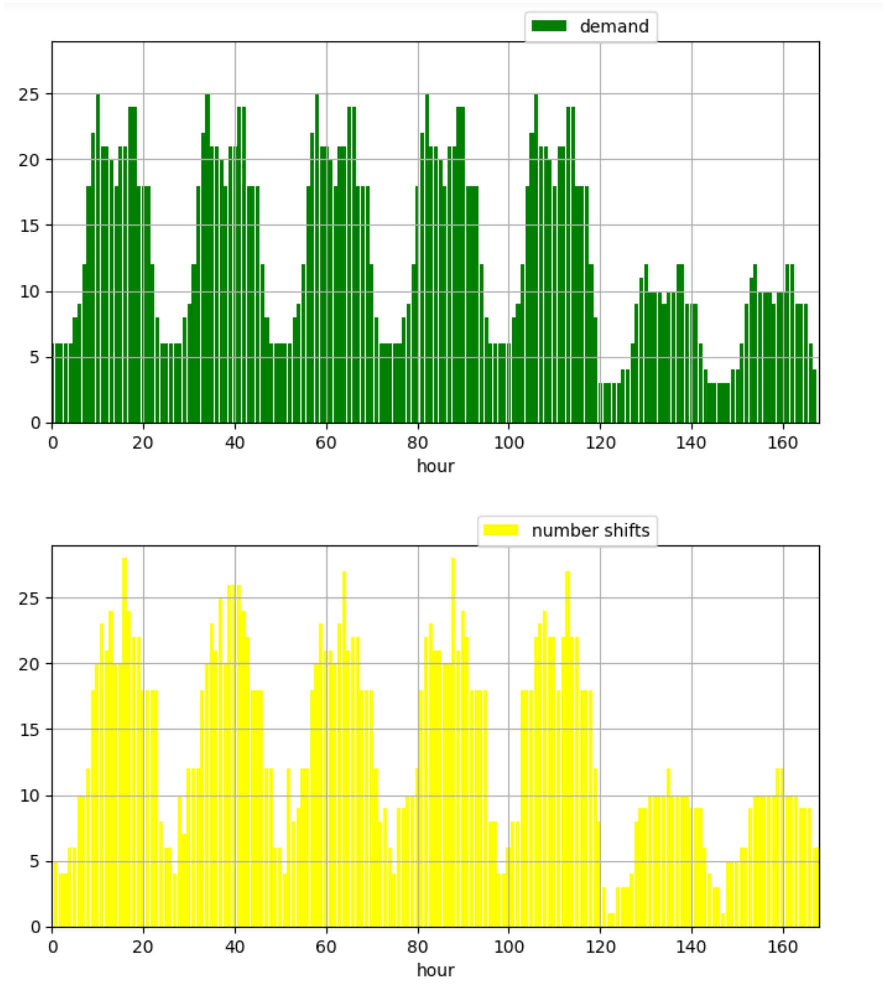

Figure 3: Demands and optimal shift coverage over the entire week.

> **A homework assignment:** Implement a program that takes the vector of weekday personnel demand **d**, weekend personnel demand **e**, and constant D describing maximum allowed understaffing as the input and outputs the optimal allocation of personnel for each hour (values of x_i variables). Given the input, construct an ILP model solving the problem described in 2. Solve the problem and output optimal values of $\mathbf{x}$ in the format specified below. Upload your source code to BRUTE, where it will be automatically evaluated.

2.1 Input/Output format

Your program will be executed with two arguments: the first one is the absolute path to the input file, and the second one is the absolute path to the output file (the output file has to be created by your program).

The input file consists of three lines. The first line contains the maximum 'under-staffed' limit of D. The second line contains 24 integers separated by space representing the vector of weekday demands **d**. The third line contains 24 integers representing the vector of weekend demands **e**.

The output consists of two lines. The first line is an integer denoting the optimal objective value. The second line contains the optimal values of $x_0, x_1, x_2, \ldots, x_{167}$ variables separated by spaces.

2.1.1 Example 1

Input:

```
6 6 6 6 6 8 9 12 18 22 25 21 21 20 18 21 21 24 24 18 18 18 12 8
3 3 3 3 3 4 4 6 9 11 12 10 10 10 9 10 10 12 12 9 9 9 6 4
2
```

Output:

```
232
0 0 0 6 0 4 0 2 6 2 3 4 3 0 0 10 2 0 3 0 3 0 0 0 0 0 1 6 0 5 0 ...
```

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall; United States Ed edition, 1993.