GVG Lab-07 Solution

Task 1. *1. Complete the homography matrix*

$$\mathtt{H} = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

that maps point with coordinates

$$\vec{u}_{1\alpha_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

from the first image into points with coordinates

$$\vec{u}_{2\alpha_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

in the second image.

2. *Find the coordinates of the point in the first image that maps into point $[2, 2]^\top$ in the second image.*

Solution:

1. From [1, Equation 8.41] we have

$$\lambda \begin{bmatrix} \vec{u}_{2\alpha_2} \\ 1 \end{bmatrix} = \mathtt{H} \begin{bmatrix} \vec{u}_{1\alpha_1} \\ 1 \end{bmatrix}, \quad \lambda \neq 0$$

$$\lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \lambda \neq 0 \quad \Leftrightarrow \quad \begin{cases} \lambda = 1 \\ 2\lambda = a + 1 \\ \lambda = 1 \\ \lambda \neq 0 \end{cases}$$

from which we get $a = 1$ and

$$\mathtt{H} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

2. Denote the unknown coordinates of the point in the first image by $[x, y]^\top$. Then similarly to the first part of the task we have

$$\lambda \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \ \lambda \neq 0 \quad \Leftrightarrow \quad \begin{cases} 2\lambda = 1 \\ 2\lambda = x + 1 \\ \lambda = x + y \\ \lambda \neq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \lambda = \frac{1}{2} \\ x = 0 \\ y = \frac{1}{2} \end{cases}$$

The coordinates then are $\begin{bmatrix} x & y \end{bmatrix}^\top = \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix}^\top$.

□

Task 2. *Points in a plane (spanned by $\vec{d}_1$ and $\vec{d}_2$) with coordinates*

$$\vec{X}_{1\delta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{X}_{3\delta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are mapped by a homography into image points with coordinates

$$\vec{u}_{1\alpha} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{u}_{2\alpha} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ \vec{u}_{3\alpha} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \vec{u}_{4\alpha} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

1. *Find a homography matrix.*
2. *Find the coordinates of the point of the plane that is mapped into point $[1, 1]^\top$ in the image.*

Solution:

1. All points in a plane have the representation

$$\vec{X}_\delta = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

If the coordinates of the projection of X to the image plane are $\vec{u}_\alpha = \begin{bmatrix} u & v \end{bmatrix}^\top$, then we are looking for a homogrpahy matrix $\mathtt{H}$ such that for each pair $(\vec{X}_\delta, \vec{u}_\alpha)$ there exists a nonzero λ that fulfills the equality

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{H} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

In other words, the scale λ may be different for each pair $(\vec{X}_\delta, \vec{u}_\alpha)$. Thus, we can reformulate algebraically the assignment as follows:

$$\lambda_i \begin{bmatrix} \vec{u}_{i\alpha} \\ 1 \end{bmatrix} = \mathtt{H} \left(\vec{X}_{i\delta} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right), \quad i = 1, \ldots, 4,\ \lambda_i \neq 0$$

We will investigate each $i = 1, \ldots, 4$ separately and see which constraints it puts on $\mathtt{H}$:

(a)

$$\lambda_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \lambda_1 \neq 0 \quad \Leftrightarrow \quad \begin{cases} h_{13} = 0 \\ h_{23} = 0 \\ h_{33} = \lambda_1 \\ \lambda_1 \neq 0 \end{cases}$$

(b)

$$\lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \lambda_2 \neq 0 \quad \Leftrightarrow \quad \begin{cases} h_{11} = 2\lambda_2 \\ h_{21} = 0 \\ h_{31} = \lambda_2 - \lambda_1 \\ \lambda_2 \neq 0 \end{cases}$$

(c)

$$\lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\lambda_2 & h_{12} & 0 \\ 0 & h_{22} & 0 \\ \lambda_2 - \lambda_1 & h_{32} & \lambda_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \lambda_3 \neq 0 \quad \Leftrightarrow \quad \begin{cases} h_{12} = 0 \\ h_{22} = \lambda_3 \\ h_{32} = \lambda_3 - \lambda_1 \\ \lambda_3 \neq 0 \end{cases}$$

(d)

$$\lambda_4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \lambda_4 \neq 0 \quad \Leftrightarrow \quad \begin{cases} \lambda_2 = \lambda_4 \\ \lambda_3 = 2\lambda_4 \\ \lambda_3 + \lambda_2 - \lambda_1 = \lambda_4 \\ \lambda_4 \neq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \lambda_2 = \lambda_4 \\ \lambda_3 = 2\lambda_4 \\ \lambda_1 = 2\lambda_4 \\ \lambda_4 \neq 0 \end{cases}$$

Hence the homography matrix has the form

$$\mathtt{H} = \lambda_4 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \quad \lambda_4 \neq 0.$$

This is no coincidence that $\mathtt{H}$ is defined up to (nonzero) scale λ_4, since according to [1, Equations 8.52-8.55], $\xi\mathtt{H}$ defines the same map as $\mathtt{H}$.

2. Denote the unknown coordinates of the point in the first image by $[x, y]^\top$. Then

$$\lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \ \lambda \neq 0 \quad \Leftrightarrow \quad \begin{cases} 2x = \lambda \\ 2y = \lambda \\ -x + 2 = \lambda \\ \lambda \neq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = \frac{2}{3} \\ y = \frac{2}{3} \\ \lambda = \frac{4}{3} \end{cases}$$

The coordinates then are $\begin{bmatrix} x & y \end{bmatrix}^\top = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top$.

□

Task 3. *Let us have a camera that captured 2 images by keeping its center fixed. Moreover, the camera cartesian coordinate system γ_2 after the motion is obtained from γ_1 before the motion by a rotation around vector $\vec{c}_2$ by the angle $\theta = 90°$. Compute the homogrphy matrix that maps the points from the first image to the second, if you know that the camera calibration matrix is*

$$\mathtt{K} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The rotation by θ around the (directed) rotation axis can be determined by the right hand rule: right 4 fingers curled in the direction of rotation and the right thumb pointing in the positive direction of the axis. The result of such a rotation is visualized in Figure 1.

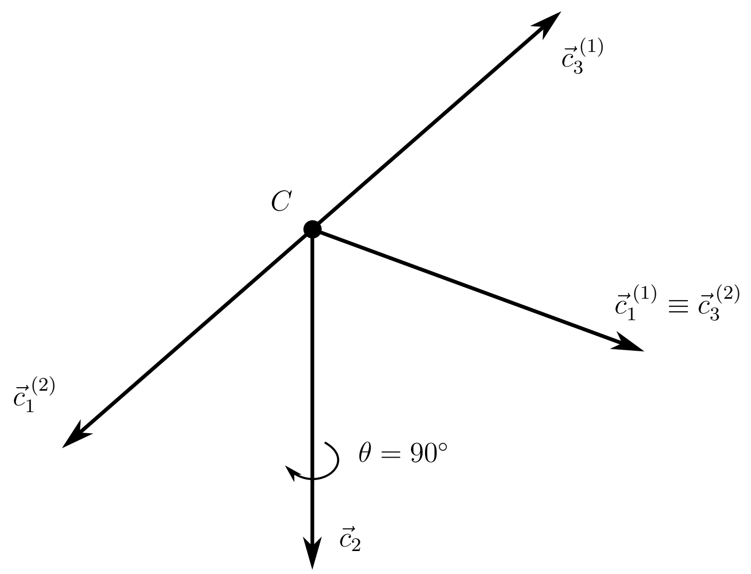

Figure 1: Two cartesian camera coordinate systems $\gamma_1 = (\vec{c}_1^{(1)}, \vec{c}_2, \vec{c}_3^{(1)})$ and $\gamma_2 = (\vec{c}_1^{(2)}, \vec{c}_2, \vec{c}_3^{(2)})$

According to [1, Equation 8.14] we have

$$\mathtt{H} = \mathtt{K}_2 \mathtt{R}_2 \mathtt{R}_1^\top \mathtt{K}_1^{-1}$$

and since in this task the camera that produced the two images is the same, then $\mathtt{K}_1 = \mathtt{K}_2 = \mathtt{K}$ and hence

$$\mathtt{H} = \underbrace{\mathtt{K}}_{\mathtt{T}_{\gamma_2 \to \beta_2}} \underbrace{\mathtt{R}_2}_{\mathtt{T}_{\kappa \to \gamma_2}} \underbrace{\mathtt{R}_1^\top}_{\mathtt{T}_{\gamma_1 \to \kappa}} \underbrace{\mathtt{K}^{-1}}_{\mathtt{T}_{\beta_1 \to \gamma_1}},$$

where

$$\mathtt{R}_2 \mathtt{R}_1^\top = \mathtt{T}_{\kappa \to \gamma_2} \mathtt{T}_{\gamma_1 \to \kappa} = \mathtt{T}_{\gamma_1 \to \gamma_2} = \begin{bmatrix} \vec{c}_{1\gamma_2}^{(1)} & \vec{c}_{2\gamma_2} & \vec{c}_{3\gamma_2}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then

$$\mathtt{H} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

□

References

[1] Tomas Pajdla, *Elements of geometry for computer vision*, `https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf`.