3 Two-view scene reconstruction

Imagine two cameras giving two images of the space from two different view points. We will next investigate how to (re-)construct camera projection matrices and meaningful coordinates of points in the space such that the reconstructed cameras and the reconstructed points generate the images.

3.1 Epipolar geometry

Figure 3.1 shows two cameras with different centers C_1, C_2 and image planes π_1, π_2, observing a general point X as u_1, u_2. Baseline b connecting
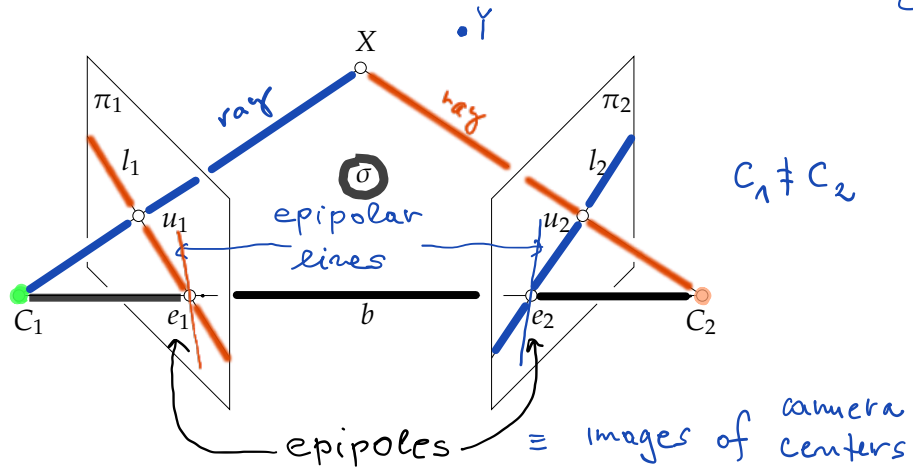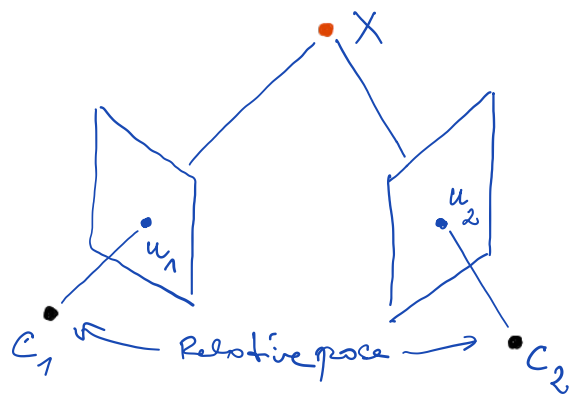

Figure 3.1: Epipolar geometry of two cameras.
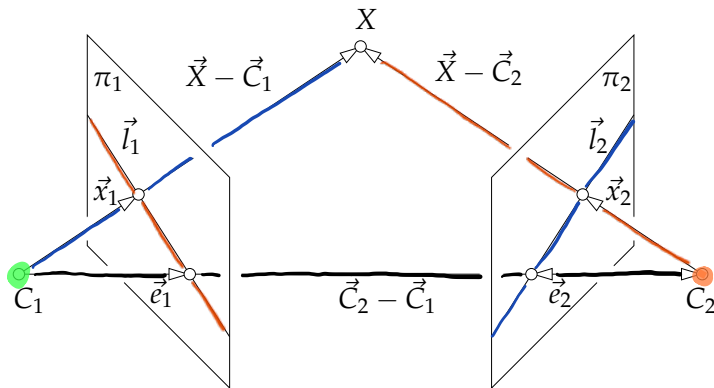
Figure 3.2: Vectors of the epipolar geometry.

image centers C_1, C_2 intersects π_1, π_2 in *epipoles* e_1, e_2. Points C_1, C_2 and X form *epipolar plane* σ, which intersects π_1 in *epipolar line* l_1 and π_2 in epipolar line l_2. Epipolar line l_1 passes through epipole e_1 and through image point u_1. Epipolar line l_2 passes through epipole e_2 and through image point u_2.

Let us next find the relationship between image points, epipoles, epipolar lines as a function of camera parameters, Figure 3.2. Assume a world coordinate system (O, δ) and cameras C_1, C_2 with camera projection matrices

$$P_1 = \left[K_1 R_1 \mid -K_1 R_1 \vec{C}_{1\delta} \right] \quad \text{and} \quad P_2 = \left[K_2 R_2 \mid -K_2 R_2 \vec{C}_{2\delta} \right] \tag{3.1}$$

Point X is projected to image planes π_1, π_2, with respective coordinate systems (o_1, β_1), (o_2, β_2), as

$$\zeta_1 \vec{x}_{1\beta_1} = P_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\beta_2} = P_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.2}$$

for some $\zeta_1 > 0$ and $\zeta_2 > 0$, which then leads to

$$\zeta_1 \vec{x}_{1\beta_1} = K_1 R_1 (\vec{X}_\delta - \vec{C}_{1\delta}) \quad \text{and} \quad \zeta_2 \vec{x}_{2\beta_2} = K_2 R_2 (\vec{X}_\delta - \vec{C}_{2\delta}) \quad (3.3)$$

$$\zeta_1 R_1^\top K_1^{-1} \vec{x}_{1\beta_1} = \vec{X}_\delta - \vec{C}_{1\delta} \qquad \zeta_2 R_2^\top K_2^{-1} \vec{x}_{2\beta_2} = \vec{X}_\delta - \vec{C}_{2\delta} \quad (3.4)$$

(handwritten top right)
$$\zeta_1 \vec{x}_{1\beta_1} = P_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\beta_2} = P_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

Consider now that vectors $\vec{X}_\delta - \vec{C}_{1\delta}$, $\vec{X}_\delta - \vec{C}_{2\delta}$ and $\vec{C}_{2\delta} - \vec{C}_{1\delta}$ form a triangle and hence

$$\vec{C}_{2\delta} - \vec{C}_{1\delta} = (\vec{X}_\delta - \vec{C}_{1\delta}) - (\vec{X}_\delta - \vec{C}_{2\delta}) \quad (3.5)$$

$$\vec{C}_{2\delta} - \vec{C}_{1\delta} = \zeta_1 R_1^\top K_1^{-1} \vec{x}_{1\beta_1} - \zeta_2 R_2^\top K_2^{-1} \vec{x}_{2\beta_2} \quad (3.6)$$

(handwritten left: $[\vec{C}_{2\delta} - \vec{C}_{1\delta}]_\times$)

with $\zeta_1 > 0$ and $\zeta_2 > 0$ for the standard choice of camera coordinate systems.

We shall next eliminate depths ζ_1, ζ_2 by exploiting the vector product identities, see Paragraph 1.3.

$$\vec{0} = \vec{x} \times \vec{x} = [\vec{x}]_\times \vec{x} \quad (3.7)$$

$$\vec{0} = \vec{y}^\top (\vec{x} \times \vec{y}) = \vec{y}^\top [\vec{x}]_\times \vec{y} \quad (3.8)$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^3$.

(handwritten: $3 \to 2$ terms)

We first vector-multiply Equation 3.6 by $\vec{C}_{2\delta} - \vec{C}_{1\delta}$ from the left to get

$$\mathbf{0} = \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times \zeta_1 R_1^\top K_1^{-1} \vec{x}_{1\beta_1} - \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times \zeta_2 R_2^\top K_2^{-1} \vec{x}_{2\beta_2} \quad (3.9)$$

and then multiply Equation 3.9 by $\zeta_2 \vec{x}_{2\beta_2}^\top K_2^{-\top} R_2$ from the left to get

(handwritten: $2 \to 1$ term)

$$0 = \zeta_2 \vec{x}_{2\beta_2}^\top K_2^{-\top} R_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times \zeta_1 R_1^\top K_1^{-1} \vec{x}_{1\beta_1} \quad (3.10)$$

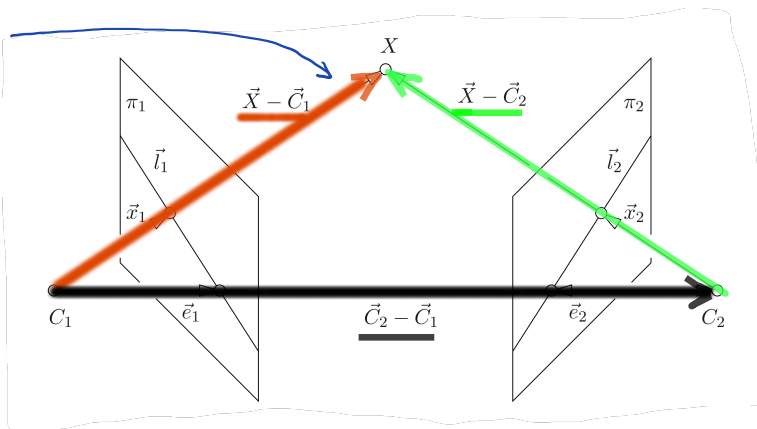which, since $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$, is equivalent with

(handwritten: elimination of ζ_1, ζ_2)

$$0 = \vec{x}_{2\beta_2}^\top K_2^{-\top} R_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times R_1^\top K_1^{-1} \vec{x}_{1\beta_1} \quad (3.11)$$

$$0 = \vec{x}_{2\beta_2}^\top K_2^{-\top} E K_1^{-1} \vec{x}_{1\beta_1} \quad (3.12)$$

$$0 = \vec{x}_{2\beta_2}^\top F \vec{x}_{1\beta_1} \quad (3.13)$$

(handwritten annotations below 3.12–3.13: $\vec{x}^\top K$ ... line; $\vec{x}_{2\beta_2}^\top \vec{x}_{2\beta_2}^* (\vec{\beta_2})$)

(handwritten right side annotations)

correspondence of points in images $\Rightarrow$ $0 = \vec{x}_{2\beta_2}^\top F \vec{x}_{1\beta_1}$

Essential matrix
$$E = R_2 [\vec{C}_{2\delta} - \vec{C}_{1\delta}]_\times R_1^\top$$

Fundamental matrix
$$F = K_2^{-\top} E K_1^{-1}$$

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where we introduced the *essential matrix* $\mathsf{E} \in \mathbb{R}^{3\times3}$ as

$$\mathsf{E} = \mathsf{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times \mathsf{R}_1^\top \tag{3.14}$$

and the *fundamental matrix* $\mathsf{F} \in \mathbb{R}^{3\times3}$ as

$$\mathsf{F} = \mathsf{K}_2^{-\top} \mathsf{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times \mathsf{R}_1^\top \mathsf{K}_1^{-1} \tag{3.15}$$

Let us next introduce epipoles to pass from vectors in δ to vectors in β_1, β_2, which are measurable in images.

The projection e_1 of the the camera center $\vec{C}_2$ to the first image as well as the projection e_2 of the the camera center $\vec{C}_1$ to the second image are obtained as

$$\zeta_1 \vec{e}_{1\beta_1} = \mathsf{P}_1 \begin{bmatrix} \vec{C}_{2\delta} \\ 1 \end{bmatrix} = \mathsf{K}_1 \mathsf{R}_1 (\vec{C}_{2\delta} - \vec{C}_{1\delta}) \tag{3.16}$$

$$\zeta_2 \vec{e}_{2\beta_2} = \mathsf{P}_2 \begin{bmatrix} \vec{C}_{1\delta} \\ 1 \end{bmatrix} = \mathsf{K}_2 \mathsf{R}_2 (\vec{C}_{1\delta} - \vec{C}_{2\delta}) \tag{3.17}$$

for some $\zeta_1 > 0$ and $\zeta_2 > 0$.

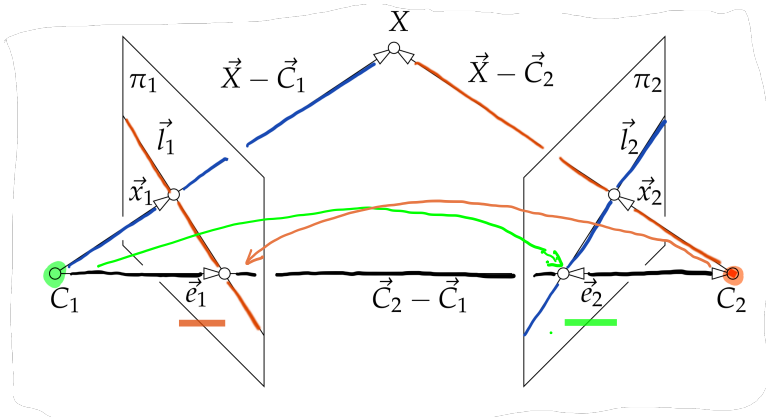We can now substitute Equation 3.16 into Equation 3.15 to get

$$\mathsf{F} = \mathsf{K}_2^{-\top} \mathsf{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times \mathsf{R}_1^\top \mathsf{K}_1^{-1} \tag{3.18}$$

$$= \mathsf{K}_2^{-\top} \mathsf{R}_2 \left[\zeta_1 \mathsf{R}_1^\top \mathsf{K}_1^{-1} \vec{e}_{1\beta_1} \right]_\times \mathsf{R}_1^\top \mathsf{K}_1^{-1} \tag{3.19}$$

$$= \zeta_1 \mathsf{K}_2^{-\top} \mathsf{R}_2 \frac{(\mathsf{R}_1^\top \mathsf{K}_1^{-1})^{-\top}}{\left| (\mathsf{R}_1^\top \mathsf{K}_1^{-1})^{-\top} \right|} \left[\vec{e}_{1\beta_1} \right]_\times \tag{3.20}$$

$$= \frac{\zeta_1}{|\mathsf{K}_1|} \mathsf{K}_2^{-\top} \mathsf{R}_2 \mathsf{R}_1^\top \mathsf{K}_1^\top \left[\vec{e}_{1\beta_1} \right]_\times \tag{3.21}$$

We used the result from §2, which shows how the vector product behaves under the change of a basis.

Analogically, we substitute Equation 3.17 into Equation 3.15 to get

$$F = K_2^{-\top} R_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times R_1^\top K_1^{-1} \tag{3.22}$$

$$= K_2^{-\top} R_2 \left[-\zeta_2 R_2^\top K_2^{-1} \vec{e}_{2\beta_2} \right]_\times R_1^\top K_1^{-1} \tag{3.23}$$
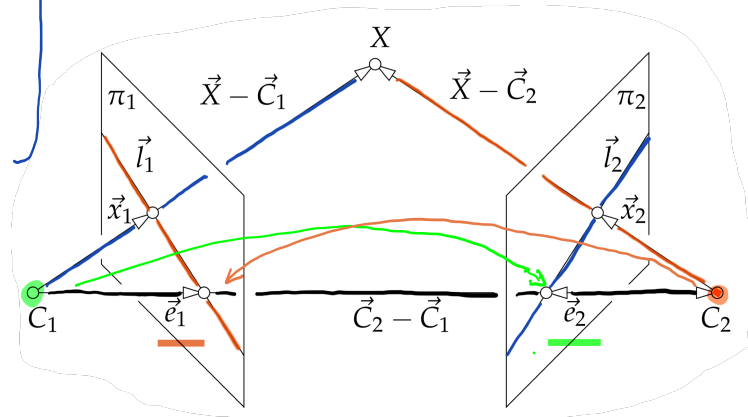
$$= \left(\left[\zeta_2 R_2^\top K_2^{-1} \vec{e}_{2\beta_2} \right]_\times R_2^\top K_2^{-1} \right)^\top R_1^\top K_1^{-1} \tag{3.24}$$

$$= \left(\frac{\zeta_2}{|K_2|} R_2^\top K_2^\top \left[\vec{e}_{2\beta_2} \right]_\times \right)^\top R_1^\top K_1^{-1} \tag{3.25}$$

$$= -\frac{\zeta_2}{|K_2|} \left[\vec{e}_{2\beta_2} \right]_\times K_2 R_2 R_1^\top K_1^{-1} \tag{3.26}$$

We used additional properties of the linear representation of the vector product from § 3.

We see from Equations 3.21 and 3.26 that it is possible to recover homogeneous coordinates of the epipoles from F by solving equations

$$F \vec{e}_{1\beta_1} = 0 \quad \text{and} \quad F^\top \vec{e}_{2\beta_2} = 0 \tag{3.27}$$

for a non-zero multiples of $\vec{e}_{1\beta_1}$, $\vec{e}_{2\beta_2}$. We also see that matrix F has rank smaller than three since it has a non-zero null space $\vec{e}_{1\beta_1}$. Since, rank of $\left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times$ is two for non-zero $\vec{C}_{2\delta} - \vec{C}_{1\delta}$, F has rank two when camera centers do not coincide.

Let us look at the epipolar lines. Epipolar lines pass through the corresponding points in images and the epipoles, i.e. $l_1 = x_1 \vee e_1$ and $l_2 x = x_2 \vee e_2$. Consider that there holds

$$\vec{x}_{2\beta_2}^\top F \vec{e}_{1\beta_1} = 0 \quad \text{and} \quad \vec{x}_{1\beta_1}^\top F^\top \vec{e}_{2\beta_2} = 0 \tag{3.28}$$

$$\vec{x}_{2\beta_2}^\top F \vec{x}_{1\beta_1} = 0 \qquad \vec{x}_{1\beta_1}^\top F^\top \vec{x}_{2\beta_2} = 0 \tag{3.29}$$

$$\tag{3.30}$$

Handwritten annotations:

NOT SUFFICIENT 4 3D Rec.

$$\vec{x}_2^\top F \vec{e} = 0$$

Analogically as above

$$F = \frac{\zeta_1}{|K_1|} K_2^{-\top} R_2 R_1^\top K_1^\top \left[\vec{e}_{1\beta_1} \right]_\times$$

$$F \vec{e}_{1\beta_1} = \frac{\zeta_1}{|K_1|} K_2^{-\top} R_2 R_1^\top K_1^\top \left[\vec{e}_{1\beta_1} \right]_\times \vec{e}_{1\beta_1} = 0$$

$$\left(\vec{e}_2^\top F = 0 \right)$$

Epipoles are null-spaces of F (E)

and therefore homogeneous coordinates $\vec{l}_{1\bar{\beta}_1}$, $\vec{l}_{2\bar{\beta}_2}$ of epipolar lines generated by $\vec{x}_{2\beta_2}$ and $\vec{x}_{1\beta_1}$, respectively, are obtained as

$$\vec{l}_{1\bar{\beta}_1} = \mathtt{F}^\top \vec{x}_{2\beta_2} \quad \text{and} \quad \vec{l}_{2\bar{\beta}_2} = \mathtt{F}\,\vec{x}_{1\beta_1} \qquad (3.31)$$

for $\vec{x}_{2\beta_2} \neq \vec{e}_{2\beta_2}$ and $\vec{x}_{1\beta_1} \neq \vec{e}_{1\beta_1}$.

(handwritten) $\vec{e}_{2\beta_2}^\top \vec{l}_{2\bar{\beta}_2} = \vec{e}_{2\beta_2}^\top \mathtt{F} \vec{x}_{1\beta_1} = 0$

3.2 Computing epipolar geometry from image matches

Let us look at how to compute the epipolar geometry between images from image matches. Our goal is to find matrix $\mathtt{G} = \tau\mathtt{F}$ for some real non-zero τ using Equation 3.13. Let us introduce

(handwritten) $|\mathtt{G}| \simeq 0$

$$\mathtt{G} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad (3.32)$$

and write Equation 3.13 as

$$0 = \vec{x}_{2\beta_2}^\top \mathtt{G}\, \vec{x}_{1i\beta_1} = \begin{bmatrix} u_{2i} & v_{2i} & w_{2i} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} &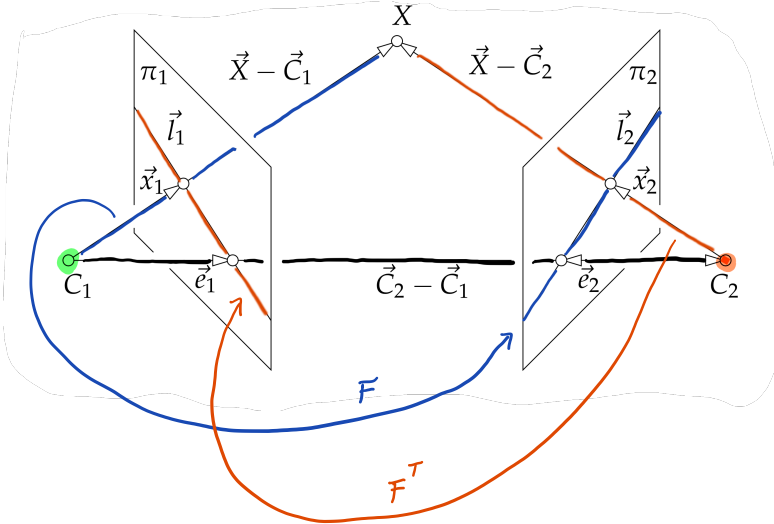 g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} u_{1i} \\ v_{1i} \\ w_{1i} \end{bmatrix} \qquad (3.33)$$

(handwritten labels) $\vec{x}_{2\beta_2}^\top$, $\mathtt{G}$, $\vec{x}_{1\beta_1}^\top$

$$0 = \begin{bmatrix} u_{2i} u_{1i} & u_{2i} v_{1i} & u_{2i} w_{1i} & v_{2i} u_{1i} & v_{2i} v_{1i} & v_{2i} w_{1i} & w_{2i} u_{1i} & w_{2i} v_{1i} & w_{2i} w_{1i} \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{12} \\ \vdots \\ g_{33} \end{bmatrix}$$

(handwritten) $w_{1i} = w_{2i} = 1$ for affine points

(handwritten) $= \begin{bmatrix} u_{1i} & v_{1i} & w_{1i} \end{bmatrix} \otimes \begin{bmatrix} u_{2i} & v_{2i} & w_{2i} \end{bmatrix} v(\mathtt{G})$
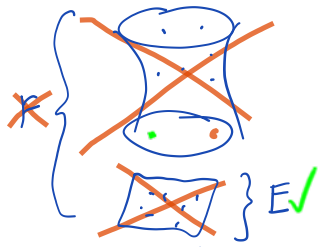
(handwritten) $\mathtt{E}$ ✓

for the i-th pair of the corresponding points $\vec{x}_{1i\beta_1}$, $\vec{x}_{2i\beta_2}$ in the two images. Notice that we can work even with ideal points when $w_{1i} = 0$ or $w_{2i} = 0$.

We can solve this way for a non-zero multiple of $\mathtt{F}$ from eight correspondences in a general position, i.e. not all on a plane or on some special

(handwritten) "Linear alg." "8-pt alg." $8 \hookrightarrow |\mathtt{G}| \neq 0$

(handwritten) In general position single plane

quadrics passing through camera centers [11]. If there is noise in image coordinates, we in general get a rank three matrix.

To avoid this problem, we can use only seven point correspondences to compute a two dimensional space of solutions

$$G = G_1 + \alpha\, G_2 \tag{3.34}$$

generated form its basis G_1, G_2 by α. Then we use the constraint

$$0 = |G| = |G_1 + \alpha\, G_2| = \left| \begin{bmatrix} g_{111} & g_{112} & g_{113} \\ g_{121} & g_{122} & g_{123} \\ g_{131} & g_{132} & g_{133} \end{bmatrix} + \alpha \begin{bmatrix} g_{211} & g_{212} & g_{213} \\ g_{221} & g_{222} & g_{223} \\ g_{231} & g_{232} & g_{233} \end{bmatrix} \right| \tag{3.35}$$

to find α by solving a third order polynomial

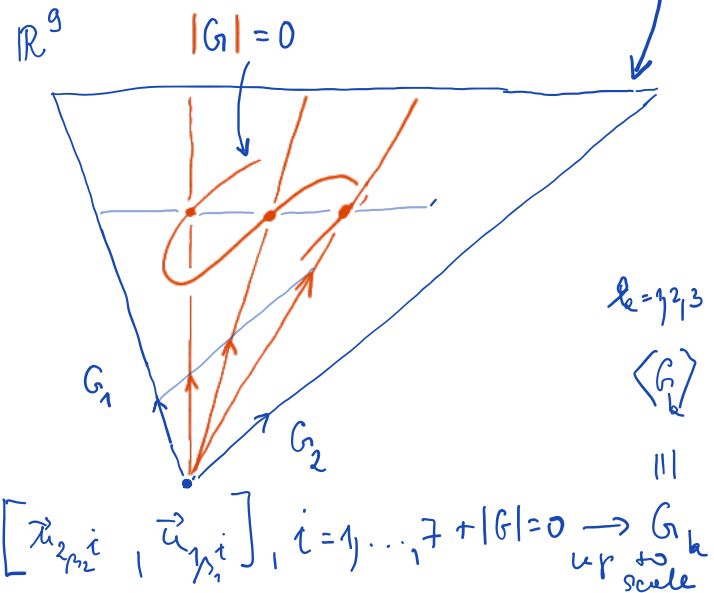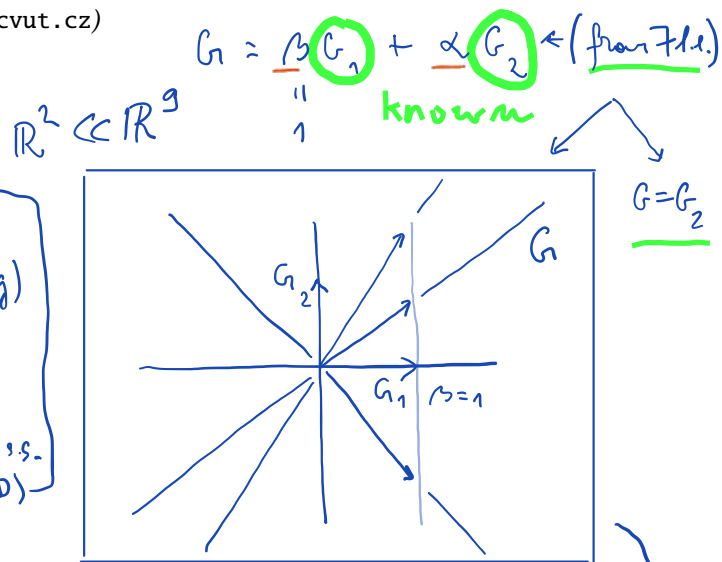$$0 = a_3\, \alpha^3 + a_2\, \alpha^2 + a_1\, \alpha + a_0 \tag{3.36}$$

$$a_3 = |G_2|$$

$$\begin{aligned}
a_2 = \ & g_{221}\, g_{232}\, g_{113} - g_{221}\, g_{212}\, g_{133} + g_{211}\, g_{222}\, g_{133} + g_{231}\, g_{112}\, g_{223} \\
& + g_{231}\, g_{212}\, g_{123} - g_{211}\, g_{223}\, g_{132} - g_{231}\, g_{122}\, g_{213} - g_{231}\, g_{222}\, g_{113} \\
& - g_{211}\, g_{123}\, g_{232} + g_{121}\, g_{232}\, g_{213} + g_{221}\, g_{132}\, g_{213} + g_{131}\, g_{212}\, g_{223} \\
& - g_{121}\, g_{212}\, g_{233} - g_{111}\, g_{223}\, g_{232} - g_{221}\, g_{112}\, g_{233} + g_{211}\, g_{122}\, g_{233} \\
& + g_{111}\, g_{222}\, g_{233} - g_{131}\, g_{222}\, g_{213}
\end{aligned}$$

$$\begin{aligned}
a_1 = \ & g_{111}\, g_{122}\, g_{233} + g_{111}\, g_{222}\, g_{133} + g_{231}\, g_{112}\, g_{123} - g_{121}\, g_{112}\, g_{233} \\
& - g_{211}\, g_{123}\, g_{132} - g_{221}\, g_{112}\, g_{133} - g_{231}\, g_{122}\, g_{113} + g_{211}\, g_{122}\, g_{133} \\
& + g_{121}\, g_{132}\, g_{213} + g_{121}\, g_{232}\, g_{113} + g_{131}\, g_{212}\, g_{123} - g_{121}\, g_{212}\, g_{133} \\
& - g_{131}\, g_{222}\, g_{113} + g_{221}\, g_{132}\, g_{113} - g_{111}\, g_{123}\, g_{232} - g_{131}\, g_{122}\, g_{213} \\
& + g_{131}\, g_{112}\, g_{223} - g_{111}\, g_{223}\, g_{132}
\end{aligned}$$

$$a_0 = |G_1|$$

That will give us up to three rank two matrices G.

Notice that we assumed that G was constructed with a non-zero coefficient at G_1. We therefore also need to check $G = G_2$ for a solution.

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Handwritten annotations:

$G = \beta G_1 + \alpha G_2 \leftarrow (\text{from 7.1.1.})$

known

$|G| = 0$

$\mathbb{R}^2 \subset\subset \mathbb{R}^9$

$8 \text{ lin} \to 7 \text{ lin.}$
$+1 \text{ Non-l.}$
$0 = A \cdot v(g)$
7×9
$2 D \cdot s.$
$3 \ 1D \ s.s. \quad 1D \ s.s. \ (SVD)$

$\deg |G| = 3$

$G = G_2$

G_2

$G \quad G_1 \ \beta = 1$

$\mathbb{R}^9 \quad |G| = 0$

$G_1 \quad G_2$

$\ell = \frac{1}{3}, 3$

$\langle \frac{G}{\ell} \rangle$

$\begin{bmatrix} \vec{u}_{2\beta_2}{}^i & \vec{u}_{\beta_1}{}^i \end{bmatrix}, \ i = 1, \dots, 7 + |G| = 0 \to G_k$ up to scale

3.3 Ambiguity in two-view reconstruction

The goal of scene reconstruction from its two views is to find camera projection matrices P_1, P_2, and coordinates of points in the scene $\vec{X}_\delta$ such that the points $\vec{X}_\delta$ are projected by cameras P_1, P_2 to observed image points $\vec{x}_{1\beta_1}$, $\vec{x}_{2\beta_2}$

$$\zeta_1 \vec{x}_{1\beta_1} = P_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\beta_2} = P_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad (3.37)$$

for some positive real ζ_1, ζ_2.

Assume that there are some cameras P_1, P_2, and coordinates of points in the scene $\vec{X}_\delta$ such that Equation 3.37 holds true. Then, for every 4×4 real regular matrix H we can get new camera matrices P'_1, P'_2 and new point coordinates $\vec{X}'_\delta$ as

$$P'_1 = P_1 H^{-1} \quad P'_2 = P_2 H^{-1} \quad \begin{bmatrix} \vec{X}'_\delta \\ 1 \end{bmatrix} = H \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad (3.38)$$

which also project to the same image points

$$\zeta_1 \vec{x}_{1\beta_1} = P_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = P_1 H^{-1} H \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = P'_1 \begin{bmatrix} \vec{X}'_\delta \\ 1 \end{bmatrix} \quad (3.39)$$

$$\zeta_2 \vec{x}_{2\beta_2} = P_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = P_2 H^{-1} H \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = P'_2 \begin{bmatrix} \vec{X}'_\delta \\ 1 \end{bmatrix} \quad (3.40)$$

We see that in general we can reconstruct the cameras and the scene points only up to some unknown transformation of the space. We also see that the transformation is more general than just changing a basis in $\mathbb{R}^3$ where we represent affine points $\vec{X}_\delta$. Matrix H acts in the three-dimensional affine space exactly as homography on two-dimensional affine space.
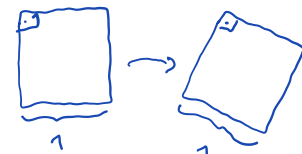
Let us next look at a somewhat simpler situation when camera calibration matrices K_1, K_2 are known. In such a case we can make sure that H
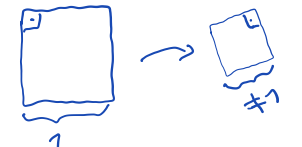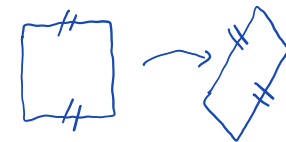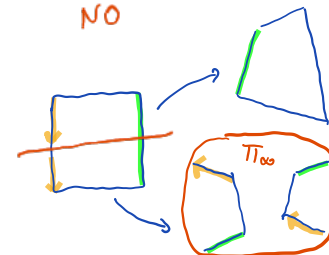
has a special form which corresponds to a special change of a coordinate system in the three-dimensional affine space.

3.4 Reconstruction from two calibrated views

Let us further assume that camera calibration matrices K_1, K_2 are known. Hence we can pass from F to E using Equations 3.14, 3.15 as

$$E = K_2^\top F K_1 \tag{3.41}$$

then recover the relative pose of the cameras, set their coordinate systems and finally reconstruct points of the scene.

3.4.1 Camera computation

To simplify the setting, we will first pass from "uncalibrated" image points $\vec{x}_{1\beta_1}$, $\vec{x}_{2\beta_2}$ using K_1, K_2 to "calibrated"

$$\vec{x}_{1\gamma_1} = K_1^{-1}\vec{x}_{1\beta_1} \quad \text{and} \quad \vec{x}_{2\gamma_2} = K_2^{-1}\vec{x}_{2\beta_2} \tag{3.42}$$

and then use camera projection matrices as follows

$$\zeta_1 \vec{x}_{1\gamma_1} = P_{1\gamma_1} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\gamma_2} = P_{2\gamma_2} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.43}$$

Matrix H allows us to choose the global coordinate system of the scene as (C_1, ϵ_1). Setting

$$H^{-1} = \begin{bmatrix} R_1^\top & \vec{C}_{1\delta} \\ \vec{0}^\top & 1 \end{bmatrix} \tag{3.44}$$

we get from Equation 3.38

$$\mathsf{P}_{1\gamma_1} = \left[\, \mathsf{I} \mid \vec{0} \,\right] \tag{3.45}$$

$$\mathsf{P}_{2\gamma_2} = \left[\, \mathsf{R}_2\,\mathsf{R}_1^\top \mid -\mathsf{R}_2\,(\vec{C}_{2\delta} - \vec{C}_{1\delta}) \,\right] = \left[\, \mathsf{R}_2\,\mathsf{R}_1^\top \mid -\mathsf{R}_2\mathsf{R}_1^\top(\vec{C}_{2\epsilon_1} - \vec{C}_{1\epsilon_1}) \,\right] \tag{3.46}$$

$$= \left[\, \mathsf{R} \mid -\mathsf{R}\,\vec{C}_{\epsilon_1} \,\right] \tag{3.47}$$

and the corresponding essential matrix

$$\mathsf{E} = \mathsf{R}\left[\vec{C}_{\epsilon_1}\right]_\times \tag{3.48}$$

From image measurements, $\vec{x}_{1\gamma_1}$, $\vec{x}_{2\gamma_2}$, we can compute, Section 3.2, matrix

$$\mathsf{G} = \tau\,\mathsf{E} = \tau\,\mathsf{R}\left[\vec{C}_{\epsilon_1}\right]_\times \tag{3.49}$$

and hence we can get E only up to a non-zero multiple τ. Therefore, we can recover $\vec{C}_{\epsilon_1}$ only up to τ.

We will next fix τ up to its sign s_1. Consider that the *Frobenius norm* of a matrix G

$$\begin{aligned}
\|\mathsf{G}\|_F &= \sqrt{\sum_{i,j=1}^{3} \mathsf{G}_{ij}^2} = \sqrt{\mathrm{trace}\,(\mathsf{G}^\top\mathsf{G})} = \sqrt{\mathrm{trace}\left(\tau^2\left[\vec{C}_{\epsilon_1}\right]_\times^\top \mathsf{R}^\top\mathsf{R}\left[\vec{C}_{\epsilon_1}\right]_\times\right)} \\
&= \sqrt{\tau^2\,\mathrm{trace}\left(\left[\vec{C}_{\epsilon_1}\right]_\times^\top\left[\vec{C}_{\epsilon_1}\right]_\times\right)} \tag{3.50} \\
&= |\tau|\,\sqrt{2\,\|\vec{C}_{\epsilon_1}\|^2} = |\tau|\,\sqrt{2}\,\|\vec{C}_{\epsilon_1}\| \tag{3.51}
\end{aligned}$$

We have used the following identities

$$\begin{aligned}
\mathsf{G}^\top\mathsf{G} &= \tau^2\left[\vec{C}_{\epsilon_1}\right]_\times^\top \mathsf{R}^\top\mathsf{R}\left[\vec{C}_{\epsilon_1}\right]_\times = \tau^2\left[\vec{C}_{\epsilon_1}\right]_\times\left[\vec{C}_{\epsilon_1}\right]_\times \tag{3.52} \\
&= \tau^2\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \tau^2\begin{bmatrix} y^2+z^2 & -x\,y & -x\,z \\ -x\,y & x^2+z^2 & = y\,z \\ -x\,z & -y\,z & x^2+y^2 \end{bmatrix}
\end{aligned}$$

We can now construct normalized matrix $\bar{\mathsf{G}}$ as

$$\bar{\mathsf{G}} = \frac{\sqrt{2}\,\mathsf{G}}{\sqrt{\sum_{i,j=1}^{3} \mathsf{G}_{ij}^2}} = \frac{\tau}{|\tau|}\,\mathsf{R}\left[\begin{array}{c} \vec{C}_{\epsilon_1} \\ \|\vec{C}_{\epsilon_1}\| \end{array}\right]_{\times} = s_1\,\mathsf{R}\left[\vec{t}_{\epsilon_1}\right]_{\times} \tag{3.53}$$

with new unknown $s_1 \in \{+1, -1\}$ and $\vec{t}_{\epsilon_1}$ denoting the unit vector in the direction of the second camera center in ϵ_1 basis.

We can find vector $\vec{v}_{\epsilon_1} = s_2\,\vec{t}_{\epsilon_1}$ with new unknown $s_2 \in \{+1, -1\}$ by solving

$$\bar{\mathsf{G}}\,\vec{v}_{\epsilon_1} = 0 \quad \text{subject to} \quad \|\vec{v}_{\epsilon_1}\| = 1 \tag{3.54}$$

to get

$$\bar{\mathsf{G}} = s_1\,\mathsf{R}\left[\frac{1}{s_2}\,\vec{v}_{\epsilon_1}\right]_{\times} = \frac{s_1}{s_2}\,\mathsf{R}\left[\vec{v}_{\epsilon_1}\right]_{\times} \tag{3.55}$$

$$s\,\bar{\mathsf{G}} = \mathsf{R}\left[\vec{v}_{\epsilon_1}\right]_{\times} \tag{3.56}$$

$$\left[\begin{array}{ccc} s\,\mathsf{g}_1 & s\,\mathsf{g}_2 & s\,\mathsf{g}_3 \end{array}\right] = \mathsf{R}\left[\begin{array}{ccc} \mathsf{v}_1 & \mathsf{v}_2 & \mathsf{v}_3 \end{array}\right] \tag{3.57}$$

with unknown $s \in \{+1, -1\}$, unknown rotation R and known matrices $\left[\begin{array}{ccc} \mathsf{g}_1 & \mathsf{g}_2 & \mathsf{g}_3 \end{array}\right] = \bar{\mathsf{G}}$ and $\left[\begin{array}{ccc} \mathsf{v}_1 & \mathsf{v}_2 & \mathsf{v}_3 \end{array}\right] = [\vec{v}_{\epsilon_1}]_{\times}$.

This is a matricial equation. Matrices $\bar{\mathsf{G}}$, $[\vec{v}_{\epsilon_1}]_{\times}$ are of rank two and hence do not determine R uniquely unless we use $\mathsf{R}^\top \mathsf{R} = \mathsf{I}$ and $|\mathsf{R}| = 1$. That leads to a set of polynomial equations. They can be solved but we will use the property of vector product, §2, to directly construct regular matrices that will determine R uniquely for a fixed s.

Consider that for every regular $\mathsf{A} \in \mathbb{R}^{3 \times 3}$, we have, §2,

$$(\mathsf{A}\,\vec{x}_\beta) \times (\mathsf{A}\,\vec{y}_\beta) = \vec{x}_{\beta'} \times \vec{y}_{\beta'} = \frac{\mathsf{A}^{-\top}}{|\mathsf{A}^{-\top}|}\,(\vec{x}_\beta \times \vec{y}_\beta) \tag{3.58}$$

which for R gives

$$(\mathsf{R}\,\vec{x}_\beta) \times (\mathsf{R}\,\vec{y}_\beta) = \mathsf{R}\,(\vec{x}_\beta \times \vec{y}_\beta) \tag{3.59}$$

Using it for $i, j = 1, 2, 3$ to get

$$
\begin{align}
(s\,\mathbf{g}_i) \times (s\,\mathbf{g}_j) &= (\mathbf{R}\,\mathbf{v}_i) \times (\mathbf{R}\,\mathbf{v}_j) \tag{3.60} \\
s^2 (\mathbf{g}_i \times \mathbf{g}_j) &= \mathbf{R}(\mathbf{v}_i \times \mathbf{v}_j) \tag{3.61} \\
(\mathbf{g}_i \times \mathbf{g}_j) &= \mathbf{R}(\mathbf{v}_i \times \mathbf{v}_j) \tag{3.62}
\end{align}
$$

i.e. three more vector equations. Notice how s disappeared in the vector product.

We see that we can write

$$
\begin{bmatrix} s\,\mathbf{g}_1 & s\,\mathbf{g}_2 & s\,\mathbf{g}_3 & \mathbf{g}_1 \times \mathbf{g}_2 & \mathbf{g}_2 \times \mathbf{g}_3 & \mathbf{g}_1 \times \mathbf{g}_3 \end{bmatrix} =
$$
$$
= \mathbf{R}_s \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \times \mathbf{v}_2 & \mathbf{v}_2 \times \mathbf{v}_3 & \mathbf{v}_1 \times \mathbf{v}_3 \end{bmatrix} \quad (3.63)
$$

There are two solutions $\mathbf{R}_+$ for $s = +1$ and $\mathbf{R}_-$ for $s = -1$. We can next compute two solutions $\vec{t}_{+\epsilon_1} = +\vec{v}_{\epsilon_1}$ and $\vec{t}_{-\epsilon_1} = -\vec{v}_{\epsilon_1}$ and combine them together to four possible solutions

$$
\begin{align}
\mathsf{P}_{2\gamma2++} &= \mathbf{R}_+ \begin{bmatrix} \mathbf{I} \,|\, -\vec{t}_{+\epsilon_1} \end{bmatrix} \tag{3.64} \\
\mathsf{P}_{2\gamma2+-} &= \mathbf{R}_+ \begin{bmatrix} \mathbf{I} \,|\, -\vec{t}_{-\epsilon_1} \end{bmatrix} \tag{3.65} \\
\mathsf{P}_{2\gamma2-+} &= \mathbf{R}_- \begin{bmatrix} \mathbf{I} \,|\, -\vec{t}_{+\epsilon_1} \end{bmatrix} \tag{3.66} \\
\mathsf{P}_{2\gamma2--} &= \mathbf{R}_- \begin{bmatrix} \mathbf{I} \,|\, -\vec{t}_{-\epsilon_1} \end{bmatrix} \tag{3.67}
\end{align}
$$

The above four camera projection matrices are compatible with $\bar{\mathsf{G}}$. The one which corresponds to the actual matrix can be selected by requiring that all reconstructed points lie in front of the cameras, i.e. that the reconstructed points are all positive multiples of vectors $\vec{x}_{1\epsilon_1}$ and $\vec{x}_{2\epsilon_2}$ for all image points.

3.4.2 Point computation

Let us assume having camera projection matrices P_1, P_2 and image points $\vec{x}_{1\beta_1}$, $\vec{x}_{2\beta_2}$ such that

$$\zeta_1 \vec{x}_{1\beta_1} = P_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\beta_2} = P_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.68}$$

We can get $\vec{X}_\delta$, and ζ_1, ζ_2 by solving the following system of (inhomogeneous) linear equations

$$\begin{bmatrix} \vec{x}_{1\beta_1} & \vec{0} & -P_1 \\ \vec{0} & \vec{x}_{2\beta_2} & -P_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ 1 \end{bmatrix} = 0 \tag{3.69}$$

3.5 Calibrated relative camera pose computation

In the previous chapter, we had first computed a multiple of the fundamental matrix from seven point correspondences and only then used camera calibration matrices to recover a multiple of the essential matrix. Here we will use the camera calibration right from the beginning to obtain a multiple of the essential matrix directly from only five image correspondences. Not only that five is smaller than seven but using the calibration right from the beginning permits all points of the scene generating the correspondences to lie in a plane.

   We start from Equation 3.42 to get $\vec{x}_{1\gamma_1}$ and $\vec{x}_{2\gamma_2}$ from Equation 3.43 which are related by

$$\vec{x}_{2\beta_2}^\top K_2^{-\top} E K_1^{-1} \vec{x}_{1\beta_1} = 0 \tag{3.70}$$

$$\vec{x}_{2\gamma_2}^\top E \vec{x}_{1\gamma_1} = 0 \tag{3.71}$$

The above equation holds true for all pairs of image points $(\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2})$ that are in correspondence, i.e. are projections of the same point of the scene.

3.5.1 **Constraints on** E

Matrix E has rank two, and therefore there holds

$$|E| = 0 \qquad (3.72)$$

true.

We will now derive additional constraints on E. Let us consider that we can write, Equation 3.48,

$$E = R\left[\vec{C}_{\epsilon_1}\right]_{\times} \qquad (3.73)$$

Let us introduce $\vec{C}_{\epsilon_1} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ and evaluate

$$
\begin{aligned}
E^\top E &= \left(R\left[\vec{C}_{\epsilon_1}\right]_{\times}\right)^\top R\left[\vec{C}_{\epsilon_1}\right]_{\times} = \left[\vec{C}_{\epsilon_1}\right]_{\times}^\top R^\top R\left[\vec{C}_{\epsilon_1}\right]_{\times} = \left[\vec{C}_{\epsilon_1}\right]_{\times}^\top \left[\vec{C}_{\epsilon_1}\right]_{\times} \quad (3.74)\\[2mm]
&= \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \begin{bmatrix} z^2 + y^2 & -xy & -xz \\ -xy & z^2 + x^2 & -yz \\ -xz & -yz & y^2 + x^2 \end{bmatrix}\\[2mm]
&= \begin{bmatrix} x^2 + y^2 + z^2 & & \\ & x^2 + y^2 + z^2 & \\ & & x^2 + y^2 + z^2 \end{bmatrix} - \begin{bmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{bmatrix}\\[2mm]
&= \|\vec{C}_{\epsilon_1}\|^2 I - \vec{C}_{\epsilon_1} \vec{C}_{\epsilon_1}^\top \qquad (3.75)
\end{aligned}
$$

We can multiply the above expression by E from the left again to get an interesting equation

$$E E^\top E = E\left(\|\vec{C}_{\epsilon_1}\|^2 I - \vec{C}_{\epsilon_1} \vec{C}_{\epsilon_1}^\top\right) = \|\vec{C}_{\epsilon_1}\|^2 E = \frac{1}{2}\text{trace}\left(E^\top E\right) E \quad (3.76)$$

or equivalently

$$2 E E^\top E = \text{trace}\left(E^\top E\right) E \qquad (3.77)$$

which provides nine equations on elements of E.

In fact, these equations also imply $|E| = 0$. Consider that Equation 3.77 implies

$$\left(2\, E\, E^\top - \mathrm{trace}\left(E^\top E\right) I\right) E = 0 \tag{3.78}$$

For Equation 3.78 to hold true, either E can't have the full rank, i.e. $|E| = 0$, or $2\, E\, E^\top - \mathrm{trace}\left(E^\top E\right) I = 0$. The latter case gives

$$0 = \mathrm{trace}\left(2\, E\, E^\top - \mathrm{trace}\left(E^\top E\right) I\right) = 2\,\mathrm{trace}\left(E\, E^\top\right) - 3\,\mathrm{trace}\left(E^\top E\right) \tag{3.79}$$

Let us check the relationship between $\mathrm{trace}\left(E^\top E\right)$ and $\mathrm{trace}\left(E\, E^\top\right)$ now. We write

$$
\begin{aligned}
\mathrm{trace}\left(E^\top E\right) &= \left(E_{11}^2 + E_{21}^2 + E_{31}^2\right) + \left(E_{12}^2 + E_{22}^2 + E_{32}^2\right) + \left(E_{13}^2 + E_{23}^2 + E_{33}^2\right) \\
&= \left(E_{11}^2 + E_{12}^2 + E_{13}^2\right) + \left(E_{21}^2 + E_{22}^2 + E_{23}^2\right) + \left(E_{31}^2 + E_{32}^2 + E_{33}^2\right) \\
&= \mathrm{trace}\left(E\, E^\top\right) \tag{3.80}
\end{aligned}
$$

Substituting the above into Equation 3.79 gets us

$$0 = 2\,\mathrm{trace}\left(E\, E^\top\right) - 3\,\mathrm{trace}\left(E^\top E\right) = -\mathrm{trace}\left(E^\top E\right) \tag{3.81}$$

Equation $2\, E\, E^\top - \mathrm{trace}\left(E^\top E\right) I = 0$ also implies

$$
\begin{aligned}
2\, E\, E^\top &= \mathrm{trace}\left(E^\top E\right) I \tag{3.82} \\
\left|2\, E\, E^\top\right| &= \left|\mathrm{trace}\left(E^\top E\right) I\right| \tag{3.83} \\
2^3 |E|^2 &= \left(\mathrm{trace}\left(E^\top E\right)\right)^3 \tag{3.84} \\
2^3 |E|^2 &= 0 \tag{3.85} \\
|E| &= 0 \tag{3.86}
\end{aligned}
$$

Therefore, Equation 3.77 implies $|E| = 0$.

Let us now look at constraints on matrix $G = \tau\, E$, for some non-zero real τ. We can multiply Equation 3.78 by τ^3 to get

$$
\begin{aligned}
\tau^3 \left(2\, E\, E^\top - \mathrm{trace}\left(E^\top E\right) I\right) E &= 0 \tag{3.87} \\
\left(2\, (\tau\, E)\, (\tau\, E^\top) - \mathrm{trace}\left((\tau\, E^\top)\, (\tau\, E)\right) I\right) (\tau\, E) &= 0 \tag{3.88} \\
\left(2\, G\, G^\top - \mathrm{trace}\left(G^\top G\right) I\right) G &= 0 \tag{3.89}
\end{aligned}
$$

Clearly, $\text{rank}(\text{G}) = \text{rank}(\tau\,\text{E}) = \text{rank}(\text{E}) = 2$.

We conclude that constraints on E and G are the same.

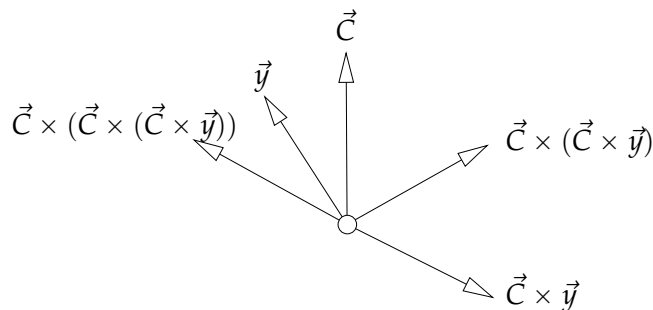### 3.5.2 Geometrical interpretation of Equation 3.77

Figure 3.3: Identity $\vec{C}_{\epsilon_1} \times (\vec{C}_{\epsilon_1} \times (\vec{C}_{\epsilon_1} \times \vec{y})) = -\|\vec{C}_{\epsilon_1}\|^2 (\vec{C}_{\epsilon_1} \times \vec{y})$.

Let us provide a geometrical interpretation of Equation 3.77. We will mutiply both sides of Equation 3.77 by a vector $\vec{y} \in \mathbb{R}^3$ and write

$$2\,\text{E}\,\text{E}^\top\text{E}\,\vec{y} = \text{trace}(\text{E}^\top\text{E})\,\text{E}\,\vec{y} \tag{3.90}$$

$$2\,\text{R}\left[\vec{C}_{\epsilon_1}\right]_\times \left[\vec{C}_{\epsilon_1}\right]_\times^\top \left[\vec{C}_{\epsilon_1}\right]_\times \vec{y} = 2\,\|\vec{C}_{\epsilon_1}\|^2\,\text{R}\left[\vec{C}_{\epsilon_1}\right]_\times \vec{y} \tag{3.91}$$

$$-\text{R}\left[\vec{C}_{\epsilon_1}\right]_\times \left[\vec{C}_{\epsilon_1}\right]_\times \left[\vec{C}_{\epsilon_1}\right]_\times \vec{y} = \text{R}\,\|\vec{C}_{\epsilon_1}\|^2\left[\vec{C}_{\epsilon_1}\right]_\times \vec{y} \tag{3.92}$$

$$\left[\vec{C}_{\epsilon_1}\right]_\times \left[\vec{C}_{\epsilon_1}\right]_\times \left[\vec{C}_{\epsilon_1}\right]_\times \vec{y} = -\|\vec{C}_{\epsilon_1}\|^2\left[\vec{C}_{\epsilon_1}\right]_\times \vec{y} \tag{3.93}$$

Now, we use that for every two vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$ there holds $[\vec{x}]_\times \vec{y} = \vec{x} \times \vec{y}$ true to get

$$\vec{C}_{\epsilon_1} \times (\vec{C}_{\epsilon_1} \times (\vec{C}_{\epsilon_1} \times \vec{y})) = -\|\vec{C}_{\epsilon_1}\|^2(\vec{C}_{\epsilon_1} \times \vec{y}) \tag{3.94}$$

which is a familiar identity of the vector pruduct in $\mathbb{R}^3$, Figure 3.3.

3.5.3 Characterization of E

Let us next see that a non-zero 3×3 real matrix satisfying Equation 3.77 has rank two and can be written in the form of Equation 3.73 for some rotation R and some vector C_{ϵ_1}.

Consider a real 3×3 matrix E such that Equation 3.77 holds true. We will make here use of the SVD decomposition [2, p. 411] of real matrices. We can write

$$E = U \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} V^\top \tag{3.95}$$

for some real non-negative a, b, c and some orthogonal real 3×3 matrices U, V, such that $U^\top U = I$, and $V^\top V = I$ [2, p. 411]. One can see that $U^\top U = I$, and $V^\top V = I$ implies $|U| = \pm 1, |V| = \pm 1$.

Using Equation 3.95 we get

$$E E^\top = U \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{bmatrix} U^\top, \quad E^\top E = V \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{bmatrix} V^\top \tag{3.96}$$

and trace $(E^\top E) =$ trace $(V D^2 V^\top) =$ trace $(V D^2 V^{-1}) =$ trace (D^2) since matrices D^2 and $E E^\top$ are similar and hence their traces, which are the sums of their eigenvalues, are equal. Now, we can rewrite Equation 3.77 as

$$\left(2 U \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{bmatrix} U^\top - (a^2 + b^2 + c^2) I \right) U \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} V^\top = \mathbf{0} \tag{3.97}$$

$$2 U \begin{bmatrix} a^3 & & \\ & b^3 & \\ & & c^3 \end{bmatrix} V^\top - (a^2 + b^2 + c^2) U \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} V^\top = \mathbf{0} \tag{3.98}$$

Matrices U, V are regular and thus we get

$$2 \begin{bmatrix} a^3 & & \\ & b^3 & \\ & & c^3 \end{bmatrix} - (a^2 + b^2 + c^2) \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} = \mathbf{0} \qquad (3.99)$$

which finally leads to the following three equations

$$
\begin{aligned}
a^3 - a\,b^2 - a\,c^2 = a\,(a^2 - b^2 - c^2) &= 0 & (3.100) \\
b^3 - b\,a^2 - b\,c^2 = b\,(b^2 - c^2 - a^2) &= 0 & (3.101) \\
c^3 - c\,a^2 - c\,b^2 = c\,(c^2 - a^2 - b^2) &= 0 & (3.102)
\end{aligned}
$$

We see that there are the following two exclusive cases:

1. If any two of a, b, c are zero, then the third one is zero too. For instance, if $a = b = 0$, then Equation 3.102 gives $c^3 = 0$. This can't happen for a non-zero E.

2. If any two of a, b, c are non-zero, then the two non-zero are equal and the third is zero. For instance, if $a \neq 0$ and $b \neq 0$, then Equations 3.100, 3.101 imply $c^2 = 0$ and thus $a^2 = b^2$, which gives $a = b$ since a, b are non-negative, i.e. rank $(E) = 2$.

We thus conclude that E can be written as

$$
\begin{aligned}
E &= U \begin{bmatrix} a & & \\ & a & \\ & & 0 \end{bmatrix} V^\top = U \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^\top & (3.103) \\
&= W \left[\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \right]_\times V^\top = W \left[V^\top V \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \right]_\times V^\top = W \frac{(V^\top)^{-\top}}{|(V^\top)^{-\top}|} \left[V \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \right]_\times & (3.104) \\
&= (\operatorname{sign}(|W|))^2 \, W\,V^\top \operatorname{sign}(|V^\top|) \, [a\,v_3]_\times & (3.105) \\
&= \operatorname{sign}(|W|) \, W\,V^\top \operatorname{sign}(|V^\top|) \, [\operatorname{sign}(|W|)\,a\,v_3]_\times & (3.106) \\
&= R\,[\operatorname{sign}(|U|)\,a\,v_3]_\times & (3.107)
\end{aligned}
$$

for some non-negative a and the third column $\mathtt{v}_3$ of $\mathtt{V}$. Parameter a is zero for $\mathtt{E} = \mathtt{0}$ and positive for rank two matrices $\mathtt{E}$. We introduced a new matrix $\mathtt{W}$ in Equation 3.104, which is the product of $\mathtt{U}$ and a rotation round the z axis. We also used $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$, and finally Equation 1.51. In Equation 3.105 we used $(\text{sign}(|\mathtt{W}|))^2 = 1$, $\mathtt{V}^{-\top} = \mathtt{V}$ for $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$. Matrix $\mathtt{R} = \text{sign}(|(\mathtt{W})|)\,\mathtt{W}\,\mathtt{V}^\top\,\text{sign}(|\mathtt{V}^\top|)$ in Equation 3.107 is a rotation since $\text{sign}(|(\mathtt{W})|)\,\mathtt{W}$ as well as $\mathtt{V}^\top\,\text{sign}(|\mathtt{V}^\top|)$ are both rotations. Finally, we see that $\text{sign}(|\mathtt{W}|) = \text{sign}(|\mathtt{U}|)$.

3.5.4 Computing a non-zero multiple of $\mathtt{E}$

Let us now disscuss how to compute a non-zero multiple of matrix $\mathtt{E}$ from image matches.

3.5.4.1 Selecting equations

Every pair of image matches $(\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2})$ provides a linear constratint on elements of $\mathtt{E}$ in the form of Equation 3.71 and matricial Equation 3.77 gives nine polynomial constraints for elements of $\mathtt{E}$.

We have already seen in Paragraph 3.2 that a non-zero multiple of $\mathtt{E}$ can be obtained from seven absolutely accurate point correspondences using the constraint $|\mathtt{E}| = 0$. The solution was obtained by solving a set of polynomial equations out of which seven were linear and the eighth one was a third order polynomial.

Let us now see how to exploit Equation 3.77 in order to compute a non-zero multiple of $\mathtt{E}$ from as few image matches as possible.

An idea might be to use Equations 3.77 instead of $|\mathtt{E}| = 0$. It would be motivated by the fact that Equations 3.77 imply equation $|\mathtt{E}| = 0$ for real 3×3 matrices $\mathtt{E}$. Unfortunately, this implication does not hold true when we allow complex numbers in $\mathtt{E}$[1], which we have to do if we want to

---

[1]Equation $|\mathtt{E}| = 0$ can't be generated from Equations 3.77 as their algebraic combination, i.e. $|\mathtt{E}| = 0$ is not in the ideal [12] generated by Equations 3.77. It means that there

obtain E as a solution to a polynomial system without using any additional constraints. We have to therefore use $|E| = 0$ as well.

The next question is whether we have to use all nine Equations 3.77. It can be shown similarly as above that indeed none of the equations 3.77 is in the ideal [12] generated by the others[2]. Therefore, we have to use all

---

might be some matrices E satisfying Equations 3.77 which do not satisfy $|E| = 0$. We know that such matrices can't be real. The proof of the above claim can be obtained by the following program in Maple [13]

```
>with(LinearAlgebra):
>with(Groebner):
>E:=¡¡e11—e12—e13¿,¡e21—e22—e23¿,¡e31—e32—e33¿¿:
>eM:=2*E.Transpose(E).E-Trace(Transpose(E).E)*E:
>eq:=expand(convert(convert(eM,Vector),list)):
>v:=indets(eq):
>mo:=tdeg(op(v)):
>G:=Basis(eq,mo):
>Reduce(Determinant(E),G,mo);
    e11 e22 e33 - e11 e23 e32 + e21 e32 e13 - e21 e12 e33 + e31 e12 e23 - e31 e22 e13
```

which computes the Groebner basis G of the ideal generated by Equations 3.77 and verifies that the remainder on division of $|E|$ by G is non-zero [12].

[2]To show that none of the equations 3.77 is in the ideal generated by the others, we run the following test in Maple.

```
>with(LinearAlgebra):
>with(Groebner):
>E:=¡¡e11—e12—e13¿,¡e21—e22—e23¿,¡e31—e32—e33¿¿:
>eM:=2*E.Transpose(E).E-Trace(Transpose(E).E)*E:
>eq:=expand(convert(convert(eM,Vector),list)):
>
>ReduceEqByEqn:=proc(eq,eqn)
    local mo,G;
    mo:=tdeg(op(indets(eqn)));
    G:=Basis(eqn,mo);
    Reduce(eq,G,mo);
  end proc:
>
>for i from 1 to 9 do
```

Equations 3.77 as well as $|E| = 0$. Hence we have altogether ten polynomial equations of order higher than one.

We have more equations than unknowns but they still do not fully determine E. We have to add some more equations from image matches. To see how many equations we have to add, we evaluate the Hilbert dimension [12] of the ideal generated by Equations 3.77 and $|E| = 0$. We know [12] that a system of polynomial equations has a finite number of solutions if and only if the Hilbert dimension of the ideal generated by the system is zero.

The Hilbert dimension of the ideal generated by Equations 3.77 and $|E| = 0$ is equal to six[3]. An extra linear equation reduces the Hilbert dimension

---

```
        ReduceEqByEqn(eq[i],eq[[op({$1..9} minus {i})]]);
    end;
```

$e11^3 + e11\,e12^2 + e11\,e13^2 + e11\,e21^2 + 2\,e21\,e12\,e22 + 2\,e21\,e13\,e23 + e11\,e31^2 + 2\,e31\,e12\,e32 + 2\,e31\,e13\,e33 - e11\,e22^2 - e11\,e32^2 - e11\,e23^2 - e11\,e33^2$

$e11^2\,e21 + 2\,e11\,e12\,e22 + 2\,e11\,e13\,e23 + e21^3 + e21\,e22^2 + e21\,e23^2 + e21\,e31^2 + 2\,e31\,e22\,e32 + 2\,e31\,e23\,e33 - e21\,e12^2 - e21\,e32^2 - e21\,e13^2 - e21\,e33^2$

$e11^2\,e31 + 2\,e11\,e12\,e32 + 2\,e11\,e13\,e33 + e21^2\,e31 + 2\,e21\,e22\,e32 + 2\,e21\,e23\,e33 + e31^3 + e31\,e32^2 + e31\,e33^2 - e31\,e12^2 - e31\,e22^2 - e31\,e13^2 - e31\,e23^2$

$e12\,e11^2 + e12^3 + e12\,e13^2 + 2\,e22\,e11\,e21 + e12\,e22^2 + 2\,e22\,e13\,e23 + 2\,e32\,e11\,e31 + e12\,e32^2 + 2\,e32\,e13\,e33 - e12\,e21^2 - e12\,e31^2 - e12\,e23^2 - e12\,e33^2$

$2\,e12\,e11\,e21 + e12^2\,e22 + 2\,e12\,e13\,e23 + e22\,e21^2 + e22^3 + e22\,e23^2 + 2\,e32\,e21\,e31 + e22\,e32^2 + 2\,e32\,e23\,e33 - e22\,e11^2 - e22\,e31^2 - e22\,e13^2 - e22\,e33^2$

$2\,e12\,e11\,e31 + e12^2\,e32 + 2\,e12\,e13\,e33 + 2\,e22\,e21\,e31 + e22^2\,e32 + 2\,e22\,e23\,e33 + e32\,e31^2 + e32^3 + e32\,e33^2 - e32\,e11^2 - e32\,e21^2 - e32\,e13^2 - e32\,e23^2$

$e13\,e11^2 + e13\,e12^2 + e13^3 + 2\,e23\,e11\,e21 + 2\,e23\,e12\,e22 + e13\,e23^2 + 2\,e33\,e11\,e31 + 2\,e33\,e12\,e32 + e13\,e33^2 - e13\,e21^2 - e13\,e31^2 - e13\,e22^2 - e13\,e32^2$

$2\,e13\,e11\,e21 + 2\,e13\,e12\,e22 + e13^2\,e23 + e23\,e21^2 + e23\,e22^2 + e23^3 + 2\,e33\,e21\,e31 + 2\,e33\,e22\,e32 + e23\,e33^2 - e23\,e11^2 - e23\,e31^2 - e23\,e12^2 - e23\,e32^2$

$2\,e13\,e11\,e31 + 2\,e13\,e12\,e32 + e13^2\,e33 + 2\,e23\,e21\,e31 + 2\,e23\,e22\,e32 + e23^2\,e33 + e33\,e31^2 + e33\,e32^2 + e33^3 - e33\,e11^2 - e33\,e21^2 - e33\,e12^2 - e33\,e22^2$

[3]The Hilber Dimension of the ideal is computed in Maple as follows

```
>with(LinearAlgebra):
>E:=¡¡e11—e12—e13¿,¡e21—e22—e23¿,¡e31—e32—e33¿¿:
>eM:=2*E.Transpose(E).E-Trace(Transpose(E).E)*E:
>eq:=expand(convert(convert(eM,Vector),list)):
>with(PolynomialIdeals):
```

by one [12]. Hence, five additional (independent) linear equations from image matches will reduce the Hilbert dimension of the system to one.

Since all equations 3.71, 3.77 and $|E| = 0$ are homogeneous, we can't reduce the Hibert dimension below one by adding more equations 3.77 from image matches. This reflects the fact that E is fixed by image measurements only up to a non-zero scale.

To conclude, five independent linear equations 3.71 plus Equations 3.77 and $|E| = 0$ fix E up to a non-zero scale.

The scale of E has to be fixed in a different way. For instance, one often knows that some of the elements of E can be set to one. By doing so, an extra independent linear equation is obtained and the Hilbert dimension is reduced to zero. Alternatively, one can ask for $\|E\|^2 = 1$, which adds a second order equation. That also reduces the Hilbert dimension to zero but doubles the number of solutions for E.

3.5.4.2 Solving the equations

We will next describe one way how to solve equations

$$\vec{x}_{i,2\gamma_2}^\top E \, \vec{x}_{i,1\gamma_1} = 0, \quad \left(2\,E\,E^\top - \operatorname{trace}\left(E^\top E\right) I\right) E = \mathbb{0}, \quad |E| = 0, \quad i = 1,\ldots,5 \tag{3.108}$$

We will present a solution based on [14], which is somewhat less efficient than [15, 16] but requires only eigenvalue computation.

>HilbertDimension(¡op(eq),Determinant(E)¿);

    6

First, using Equation 1.90 from Paragraph 1.5, we can write

$$
\begin{bmatrix}
\vec{x}_{1,1\gamma_1}^\top \otimes \vec{x}_{1,2\gamma_2}^\top \\
\vec{x}_{2,1\gamma_1}^\top \otimes \vec{x}_{2,2\gamma_2}^\top \\
\vec{x}_{3,1\gamma_1}^\top \otimes \vec{x}_{3,2\gamma_2}^\top \\
\vec{x}_{4,1\gamma_1}^\top \otimes \vec{x}_{4,2\gamma_2}^\top \\
\vec{x}_{5,1\gamma_1}^\top \otimes \vec{x}_{5,2\gamma_2}^\top \\
\vec{a}^\top
\end{bmatrix}
v(\mathsf{E}) =
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1
\end{bmatrix}
\tag{3.109}
$$

to obtain a 6×9 matrix of a system of linear equations on $v(\mathsf{E})$. Row $\vec{a}^\top$ can be chosen randomly to fix the scale of $v(\mathsf{E})$. There is only a negligible chance that it will be chosen in the orthogonal complement of the span of the solutions to force the solutions be trivial. If so, it can be detected and a new $\vec{a}^\top$ generated.

Assuming that the rows of the matrix of the system are linearly independent, we obtain a 3-dimensional affine space of solutions. After rearranging the particular solution, resp. the basis of the solution of the associated homogeneous system, back to 3×3 matrices G_0, resp. $\mathsf{G}_1, \mathsf{G}_2, \mathsf{G}_3$, we will get all solutions compatible with Equation 3.109 in the form

$$
\mathsf{G} = \mathsf{G}_0 + x\,\mathsf{G}_1 + y\,\mathsf{G}_2 + z\,\mathsf{G}_3 \tag{3.110}
$$

for $x, y, z \in \mathbb{R}$.

Now, we can substitute G for E into the two remaining equations in 3.108. We get ten trird-order polynomial equations in three unknowns and with 20 monomials. We can write it as

$$
\mathsf{M}\,\mathsf{m} = 0 \tag{3.111}
$$

where M is a constant 10×20 matrix[4] and

$$
\mathsf{m}^\top = [x^3, y\,x^2, y^2 x, y^3, z\,x^2, z\,y\,x, z\,y^2, z^2 x, z^2 y, z^3, x^2, y\,x, y^2, z\,x, z\,y, z^2, x, y, z, 1]
\tag{3.112}
$$

---

[4]Matrix M can be obtained by the following Maple [13] program

is a vector of 20 monomials.

Next, we rewrite the system 3.112 as

$$(z^3 C_3 + z^2 C_2 + z\, C_1 + C_0)\, c \;=\; 0 \qquad (3.113)$$

with

$$C \;=\; z^3 C_3 + z^2 C_2 + z\, C_1 + C_0 \qquad (3.114)$$

containing 10 monomials. Matrices $C_0, \ldots, C_4$ are constant 10×10 matrices

$$
\begin{aligned}
C_0 &= \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_{11} & m_{12} & m_{13} & m_{17} & m_{18} & m_{20} \end{bmatrix} & (3.115) \\
C_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & m_5 & m_6 & m_7 & m_{14} & m_{15} & m_{19} \end{bmatrix} & (3.116) \\
C_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_8 & m_9 & m_{16} \end{bmatrix} & (3.117) \\
C_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{10} \end{bmatrix} & (3.118)
\end{aligned}
$$

where m_i are columns of M.

Since m contains all monomials in x, y, z up to degree three, we could have written similar equations as Equation 3.113 with x and y.

```
>with(LinearAlgebra):
>G0:=¡¡g011—g012—g013¿,¡g021—g022—g023¿,¡g031—g032—g033¿¿:
>G1:=¡¡g111—g112—g113¿,¡g121—g122—g123¿,¡g131—g132—g133¿¿:
>G2:=¡¡g211—g212—g213¿,¡g221—g222—g223¿,¡g231—g232—g233¿¿:
>G3:=¡¡g311—g312—g313¿,¡g321—g322—g323¿,¡g331—g332—g333¿¿:
>trc:=E-¿simplify((2*E.Transpose(E)-Trace(Transpose(E).E)*IdentityMatrix(3,3)).E):
>eq:=[op(convert(trc(G),listlist)),Determinant(G)]:
>mo:=tdeg(x,y,z);
>m:=PolyVarMonomials(eq,mo);
```
$m := [x^3, y\,x^2, y^2 x, y^3, z\,x^2, z\,y\,x, z\,y^2, z^2 x, z^2 y, z^3, x^2, y\,x, y^2, z\,x, z\,y, z^2, x, y, z, 1]$
```
>M:=PolyCoeffMatrix(eq,m,mo):
>M[1,1];
```
$2\,g122\,g112\,g121 + 2\,g133\,g113\,g131 - g123^2\,g111 - g122^2\,g111 + 2\,g132\,g112\,g131 - g132^2\,g111 + g131^2\,g111 + g112^2\,g111 + g111^3 + 2\,g123\,g113\,g121 - g133^2\,g111 + g121^2\,g111 + g113^2\,g111$

Equation 3.113 is known as a Polynomial Eigenvealue Problem (PEP) [17] of degree three. The strandard solution to such a problem is to relax it into a generelized eigenvalue problem of a larger size as follows.

We can write $z^2\mathsf{c} = z(z\mathsf{c})$ and $z\mathsf{c} = z(\mathsf{c})$ altogether with Equation 3.113 in a matrix form as

$$
\begin{bmatrix} \mathbb{0} & \mathtt{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathtt{I} \\ -\mathsf{C}_0 & -\mathsf{C}_1 & -\mathsf{C}_2 \end{bmatrix} \begin{bmatrix} \mathsf{c} \\ z\mathsf{c} \\ z^2\mathsf{c} \end{bmatrix} = z \begin{bmatrix} \mathtt{I} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathtt{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathsf{C}_3 \end{bmatrix} \begin{bmatrix} \mathsf{c} \\ z\mathsf{c} \\ z^2\mathsf{c} \end{bmatrix} \qquad (3.119)
$$

$$
\mathsf{A}\,\mathsf{v} = z\,\mathsf{B}\,\mathsf{v} \qquad (3.120)
$$

This is a Generelized Eigenvalue Problem (GEP) [17] of size 30×30, which can be solved for z and v. Values of x, y can be recovered from v as $x = \mathsf{c}_8/\mathsf{c}_{10}$ and $x = \mathsf{c}_9/\mathsf{c}_{10}$. It provides 30 solutions in general.

When C_0 is regular, we can pass to a standard eigenvalue problem for a non-zero z by inverting A and using $w = 1/z$

$$
\begin{bmatrix} -\mathsf{C}_0^{-1}\mathsf{C}_1 & -\mathsf{C}_0^{-1}\mathsf{C}_2 & -\mathsf{C}_0^{-1}\mathsf{C}_3 \\ \mathtt{I} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathtt{I} & \mathbb{0} \end{bmatrix} \begin{bmatrix} w^2\mathsf{c} \\ w\mathsf{c} \\ \mathsf{c} \end{bmatrix} = w \begin{bmatrix} w^2\mathsf{c} \\ w\mathsf{c} \\ \mathsf{c} \end{bmatrix} \qquad (3.121)
$$