# Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

# 2021 Lecture 7

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# 8 Projective plane

## 8.1 Motivation – perspective projection in affine space

§ **1 Geometric model of perspective projection in affine space** The perspective projection of a point $X$ by a camera with projection center $C$ can be obtained geometrically in 3D affine space by taking all lines passing through the points $C$ and $X$ and finding the intersections with the (affine) image plane $\pi$.

Three different situations may arise, Figure 8.1.

1. If $X = C$, then there is an infinite number of lines passing through $C = X$, which intersect $\pi$ in all its points, and therefore the projection of $X$ contains the whole plane $\pi$.

2. If point $Y \neq C$ lies in the plane $\sigma$, which is parallel to $\pi$ and passing through $C$, then the line passing trough $C$ and $Y$ (which there is exactly one) *does not intersect* the projection plane $\pi$, and therefore, the projection of $X$ is empty.

3. If $X$ does not lie in the plane $\sigma$, then there is exactly one line passing through points $C$ and $X$ and this line intersects the projection plane $\pi$ in exactly one point $x$. Hence the projection of $X$ contains exactly one point $x$.

Let us compare this affine geometrical model of the perspective projection with the algebraic model of the perspective projection, which we have developed before.

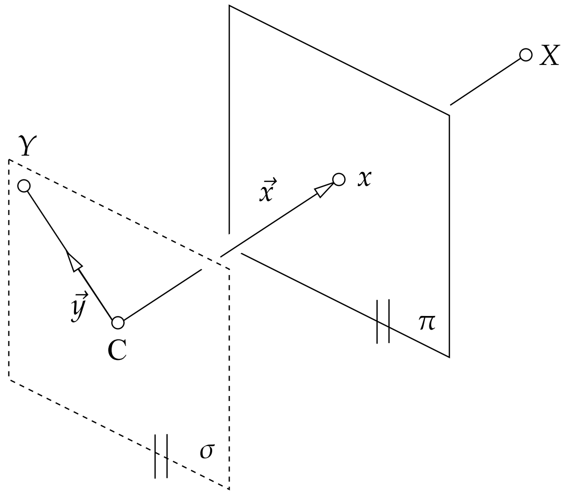


Figure 8.1: Geometric model of perspective projection in affine space. Point $C$ has infinite (i.e. not unique) projection, point $X$ has exactly one projection $x$. Point $Y$ has no projection.

**§ 2 Algebraic model of perspective projection in affine space** The projection $\vec{x}_\beta$ of $\vec{X}_\delta$ by a perspective camera with image projection matrix

$$\mathtt{P}_\beta = \begin{bmatrix} \mathtt{A} \,|\, -\mathtt{A}\,\vec{C}_\delta \end{bmatrix} \tag{8.1}$$

is

$$\eta\,\vec{x}_\beta = \begin{bmatrix} \mathtt{A} \,|\, -\mathtt{A}\,\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{8.2}$$

for some $\eta \in \mathbb{R}$.

We shall analyze the three situations, which arise with the geometrical model of affine projection.

1. If $X = C$, then
$$\eta\,\vec{x}_\beta = \begin{bmatrix} \mathtt{A} \,|\, -\mathtt{A}\,\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \vec{0} \tag{8.3}$$
i.e. we obtain the zero vector. What does it say about $\vec{x}_\beta$? Clearly, $\vec{x}_\beta$ can be *completely arbitrary* (even the zero vector) when we set $\eta = 0$. Alternatively, we can choose $\eta \neq 0$ and thus enforce $\vec{x}_\beta = \vec{0}$. Both choices are possible. We shall use the latter one since we will see that it better fits the other cases. We will use $\vec{x}_\beta = \vec{0}$ to (somewhat strangely) represent all non-zero vectors in $\mathbb{R}^3$.

2. If point $Y \neq C$ lies in the plane $\sigma$, then
$$\eta\,\vec{x}_\beta = \begin{bmatrix} \mathtt{A} \,|\, -\mathtt{A}\,\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{Y}_\delta \\ 1 \end{bmatrix} = \mathtt{A}\,(\vec{Y}_\delta - \vec{C}_\delta) \tag{8.4}$$
which, taking into account rank $\mathtt{A} = 3$, implies
$$\eta\,\mathtt{A}^{-1}\vec{x}_\beta = \vec{Y}_\delta - \vec{C}_\delta \tag{8.5}$$
Matrix $\mathtt{A}^{-1}$ transforms $\vec{x}_\beta$ into $\vec{x}_\delta$ and therefore its columns
$$\mathtt{A}^{-1} = \begin{bmatrix} \vec{b}_{1\delta} & \vec{b}_{2\delta} & \vec{b}_{3\delta} \end{bmatrix} \tag{8.6}$$
are the basic vectors of the camera coordinate system in the world basis $\delta$. Hence
$$\eta \begin{bmatrix} \vec{b}_{1\delta} & \vec{b}_{2\delta} & \vec{b}_{3\delta} \end{bmatrix} \vec{x}_\beta = \vec{Y}_\delta - \vec{C}_\delta \tag{8.7}$$
which means that vector $\vec{Y}_\delta - \vec{C}_\delta$ can be written as a linear combination of the camera coordinate system basic vectors
$$\eta\,p\,\vec{b}_{1\delta} + \eta\,q\,\vec{b}_{2\delta} + \eta\,r\,\vec{b}_{3\delta} = \vec{Y}_\delta - \vec{C}_\delta \tag{8.8}$$

with $p, q, r \in \mathbb{R}$. Now, since $Y$ lies in a plane parallel to $\pi$, vector $\vec{Y}_\delta - \vec{C}_\delta$ can be written as a linear combination of the first two basic vectors of the camera coordinate system, and therefore $r = 0$, i.e.

$$\vec{x}_\beta = \begin{bmatrix} p \\ q \\ 0 \end{bmatrix} \tag{8.9}$$

We also see that $\eta \neq 0$ since otherwise we would get the zero vector on the left but that would correspond to $Y = C$, which we have excluded.

3. If $X$ does not lie in the plane $\sigma$, then the coefficient $r \in \mathbb{R}$ in the linear combination

$$\eta\, \mathtt{A}^{-1}\, \vec{x}_\beta = \vec{X}_\delta - \vec{C}_\delta \tag{8.10}$$

$$\eta\, p\, \vec{b}_{1\delta} + \eta\, q\, \vec{b}_{2\delta} + \eta\, r\, \vec{b}_{3\delta} = \vec{X}_\delta - \vec{C}_\delta \tag{8.11}$$

is non-zero. In that case we can write

$$\eta \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{8.12}$$

$$(\eta\, r) \begin{bmatrix} \frac{p}{r} \\ \frac{q}{r} \\ 1 \end{bmatrix} = \mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{8.13}$$

$$\eta' \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{8.14}$$

As in the case two, $\eta \neq 0$ since otherwise we would get the zero vector on the left and that would correspond to $X = C$, which we have excluded.

Table 8.1: Comparison of the geometrical and algebraic projection models in affine space.

| Point position | Projection | |
| --- | --- | --- |
| | Geometrical model in aff. space | Algebraic model in aff. space |
| $X \notin \sigma$ | one point of $\pi$ | $\eta \neq 0, \vec{x}_\beta = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}, (\vec{x}_\beta \neq \vec{0})$ |
| $C \neq X \in \sigma$ | no point | $\eta \neq 0, \vec{x}_\beta = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \vec{x}_\beta \neq \vec{0}$ |
| $X = C$ | all points of $\pi$ | $\eta \neq 0, \vec{x}_\beta = \vec{0}$ |

The comparison of the two models of perspective projection, Table 8.1 shows that

1. We can always assume $\eta \neq 0$.
2. The "projection" of $C$ is represented by the zero vector while the projections of all other points are represented by non-zero vectors.
3. The algebraic projection model can represent projections of all points in the affine space.
4. The geometrical projection model is less capable than the algebraic projection model since it can't model the projection of points in $\sigma$ different from $C$.

The previous analysis clearly shows that the affine geometrical model of the perspective projection is somewhat deficient. It can't model projections of some points in the space. This deficiency leads to inventing a generalized model of the geometry around us in order to model the perspective projection completely by intersections of geometrical entities. This generalization of the affine space is called the *projective space*.

Let us look at the most important projective space, which is the *projective plane*. We shall first develop a concrete projective plane by improving the affine plane exactly as much as necessary to achieve what we want, i.e. to be able to distinguish projections of all points in the space. In fact, this will be extremely easy since we have already done all the work, and we only need to "upgrade" the notion of point, line, intersection and join (i.e. making the line from two distinct points). Later, we shall observe that such an "upgrade" will also lead to an interesting simplification and generalization of the principles of geometry.

## 8.2 Real projective plane

### 8.2.1 Geometrical model of the real projective plane

A real affine plane $\mathcal{A}^2$ can be imagined as a subset of a real affine space $\mathcal{A}^3$, Figure 8.2. There is a point $O$ in $\mathcal{A}^3$, which is not in $\mathcal{A}^2$. For each point $X$ in $\mathcal{A}^2$, there is exactly one line in $\mathcal{A}^3$, which passes through $X$ and $O$. Now, there is a set of lines in $\mathcal{A}^3$, which pass through $O$ but do not pass through any point of $\mathcal{A}^2$. This is the set of lines parallel to $\mathcal{A}^2$ that pass through $O$. These lines fill the plane of $\mathcal{A}^3$, which is parallel to $\mathcal{A}^2$ and which contains the point $O$.

The set of all lines in $\mathcal{A}^3$ passing through $O$ will be called the *real projective plane* and denoted as $\mathbb{P}^2$. The lines of $\mathcal{A}^3$ passing through $O$ will be called the *points of the real projective plane*.[1]

[1] The previous definition can be given without referring to any affine plane. We can take a point $O$ in $\mathcal{A}^3$ and the set of all lines in $\mathcal{A}^3$ passing through $O$ and call it a projective plane. In the above example, however, we have obtained the projective plane as an extension of a given affine plane $\mathcal{A}^2$. In such a case, we can distinguish two sets of points – affine points and ideal points – in the projective plane.

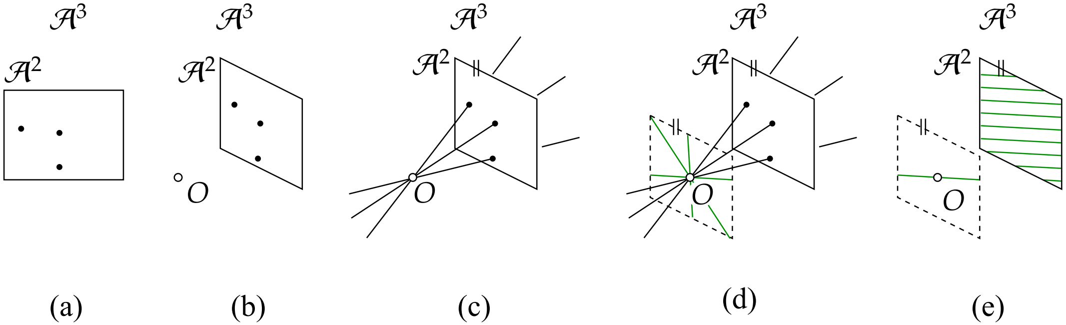


Figure 8.2: (a) Two dimensional affine plane $\mathcal{A}^2$ can be (b) embedded in the three dimensional affine space $\mathcal{A}^3$. There is a point $O \in \mathcal{A}^3$, $O \notin A^2$. (c) For each point $X$ in $\mathcal{A}^2$, there is exactly one line through $X$ and $O$ in $\mathcal{A}^3$. (d) There is exactly one pencil of lines through $O$, which do not correspond to any point in $\mathcal{A}^2$, in $\mathcal{A}^3$. (e) Each line in the pencil corresponds to a set of parallel lines with the same direction in $\mathcal{A}^2$.

The lines in $\mathcal{A}^3$ passing through $O$, which intersect $\mathcal{A}^2$, are in one-to-one correspondence with points in the affine plane $\mathcal{A}^2$ and hence will be called the *affine points of the projective plane*[2] of the projective plane. The set of lines in $\mathcal{A}^3$ passing through $O$, which do not intersect $\mathcal{A}^2$, are the "additional" points of the projective plane and will be called the *ideal points of the projective plane*[3],[4]

[2]Vlastní body in Czech. Finite points in [13].
[3]Nevlastní body in Czech. Points at infinity in [13].
[4]Notice that words "point" and "line" actually need to be accompanied by adjectives for the above to make sense beacause lines of $\mathcal{A}^3$ become points of $\mathcal{A}^2$. Also notice that this division of the points of the projective plane makes sense only when we start with a given affine plane or when we start with a projective plane and select one plane of lines in $\mathcal{A}^3$ as the set of ideal points.

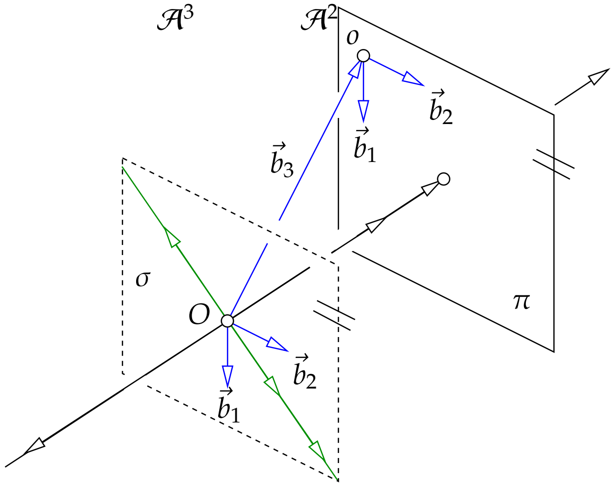


Figure 8.3: Algebraic model of the real projective plane.

To each ideal point $P$ (i.e. a line $l$ of $\mathcal{A}^3$ through $O$ parallel to $\mathcal{A}^2$), there corresponds exactly one set of parallel lines in $\mathcal{A}^2$ which are parallel to $l$ in $A^3$. Different sets of parallel lines in $\mathcal{A}^2$ are distinguished by their direction. In this sense, ideal points correspond to directions in $\mathcal{A}^2$ and can also be understood as points where parallel lines of $\mathcal{A}^2$ intersect. Notice that the parallel lines of $A^2$ do not intersect in $\mathcal{A}^2$, because $P$ is not in $\mathcal{A}^2$, but they intersect in the real projective plane obtained as the extension of $\mathcal{A}^2$.

### 8.2.2 Algebraic model of the real projective plane

We shall now move from the geometrical model in $\mathcal{A}^3$ to an algebraic model in $\mathbb{R}^3$ which allows us to do computations.

Let us choose a coordinate system $(O, \vec{b}_1, \vec{b}_2, \vec{b}_3)$ in $\mathcal{A}^3$ with the origin in $O$, with basic vectors $\vec{b}_1, \vec{b}_2$ from the coordinate system $(o, \vec{b}_1, \vec{b}_2)$ in $\mathcal{A}^2$ and with $\vec{b}_3 = \varphi(O, o)$, Figure 8.3.

Lines in $\mathcal{A}^3$, which pass through $O$, correspond to one-dimensional subspaces of $\mathbb{R}^3$ and therefore, in $\mathbb{R}^3$, points of the real projective plane will be represented by one-dimensional subspaces.

The *real projective plane* is the set of all one-dimensional subspaces of $\mathbb{R}^3$.

The affine plane is a subset of the set of all one-dimensional subspaces of $\mathbb{R}^3$, which we obtain after removing all one-dimensional subspaces that lie in a two-dimensional subspace of $\mathbb{R}^3$.

There are (infinitely) many possible choices of sets of one-dimensional subspaces which can model the affine plane within the real projective plane. The choice of a particular subset, which will model a concrete1 affine plane, can be realized by a choice of a basis in $\mathbb{R}^3$.

Let us select a basis $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ of $\mathbb{R}^3$. Then, all the one-dimensional subspaces generated by vectors

$$\vec{x}_\beta = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad x, y \in \mathbb{R} \tag{8.15}$$

will represent affine points, point $X$ in Figure 8.4, and all the one-dimensional subspaces generated by vectors

$$\vec{x}_\beta = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad x, y \in \mathbb{R},\ x \neq 0 \text{ or } y \neq 0 \tag{8.16}$$

will represent the ideal points, e.g. point $Y$ in Figure 8.4.

It is clear that the affine points are in one-to-one correspondence with all points in a two-dimensional affine space (plane) and the ideal points are exactly what we need to add to the affine points to get all one-dimensional subspaces of $\mathbb{R}^3$.

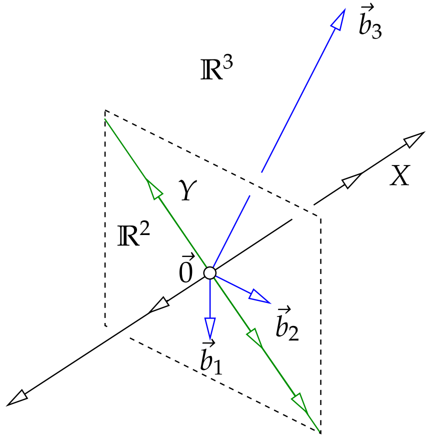


Figure 8.4: Points of the real projective plane are represented by one-dimensional subspaces of $\mathbb{R}^3$. One selected two-dimensional affine subspace determines the ideal points.

### 8.2.3 Lines of the real projective plane

Let us look at lines now. Lines, e.g. $l$ in Figure 8.5, in the affine plane contain points represented by one-dimensional subspaces generated, e.g., by $\vec{x}$ and $\vec{y}$. This set of one-dimensional subspaces of points on $l$ fills almost a complete two-dimensional subspace of $\mathbb{R}^3$ with the exception of one one-dimensional subspace, generated by $\vec{z}$, which represents an ideal point. After adding the subspace generated by $\vec{z}$ to the set of all one-dimensional subspaces representing points on $l$, we completely fill a two-dimensional subspace of $\mathbb{R}^3$, which hence corresponds to the *projective completion of the affine line* $l$, which we will further call line, too.

Hence, in the real projective plane, *lines correspond to two-dimensional subspaces of* $\mathbb{R}^3$.

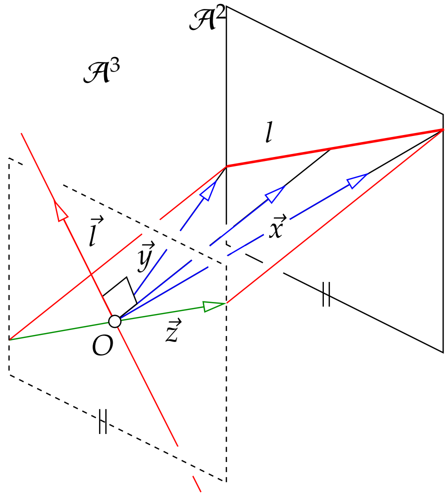


Figure 8.5: Lines of the real projective plane correspond to two-dimensional subspaces of $\mathbb{R}^3$ but can be also represented by one-dimensional subspaces of $\mathbb{R}^3$.

We would like to do calculations with lines as we do calculations with points. Let us develop a convenient representation of lines now. A straightforward way how to represent a two-dimensional subspace in $\mathbb{R}^3$ is to select a basis (i.e. two linearly independent vectors) of the subspace, e.g. $\vec{x}$ and $\vec{y}$ for the line $l$. There are many ways how to choose a basis and therefore the representation is far from unique. Moreover, having two bases, it is not apparent whether they represent the same subspace.

For instance, two pairs of linearly independent vectors $(\vec{x}_1, \vec{y}_1)$ and $(\vec{x}_2, \vec{y}_2)$ represent the same line if and only if they generate the same two-dimensional subspace. To verify that, we, for instance, may check whether

$$\operatorname{rank}\left[\vec{x}_{1\beta} \quad \vec{y}_{1\beta} \quad \vec{x}_{2\beta} \quad \vec{y}_{2\beta}\right] = 2 \tag{8.17}$$

where we write all the four vectors $\vec{x}_1$, $\vec{y}_1$, $\vec{x}_2$, $\vec{y}_2$ w.r.t. a basis $\beta$ of $\mathbb{R}^3$.

Yet, there is another quite convenient way how to represent a two dimensional subspace in $\mathbb{R}^3$. Since $3 = 2 + 1$, we can find for each two-dimensional subspace, specified by a basis $(\vec{x}, \vec{y})$, exactly one one-dimensional subspace of the three-dimensional dual linear space. Call the basis of this new one-dimensional subspace $\vec{l}$. Then there holds

$$\vec{l}_{\bar{\beta}}^{\top} \begin{bmatrix} \vec{x}_\beta & \vec{y}_\beta \end{bmatrix} = 0 \tag{8.18}$$

where $\bar{\beta}$ is the dual basis to $\beta$. Therefore, we can represent lines in the real projective plane by one-dimensional subspaces in this way.

We have developed an interesting representation of points and lines where both points and lines are represented by one-dimensional subspaces of $\mathbb{R}^3$. Points are represented by one-dimensional subspaces of $V = \mathbb{R}^3$, which is connected by $\varphi$ to the three-dimensional space $\mathcal{A}^3$ of the geometrical model of the real projective plane. The lines are represented by one-dimensional subspaces of the space $\bar{V}$, which is the space dual to $V$. Using the basis $\bar{\beta}$ in $\bar{V}$, which is dual to basis $\beta$ in $V$, the coordinates $\vec{l}_{\bar{\beta}}$ as well as coordinates of $\vec{x}_\beta$ become vectors in $\mathbb{R}^3$ which satisfy Equation 8.18.

The line of $\mathcal{A}^3$ generated by $\vec{l}$ in Figure 8.5 is shown as perpendicular[5] to the plane generated by $\vec{x}$, $\vec{y}$. Indeed, in the geometrical model of the real projective plane, we can use the notion of perpendicularity to uniquely construct the (perpendicular) line to the plane corresponding to $l$ in $\mathcal{A}^2$.

### 8.2.4 Ideal line

The set of all one-dimensional subspaces of $\mathbb{R}^3$, which do not correspond to points in the affine plane, i.e. the set of all ideal points, forms itself a two-dimensional subspace of $\mathbb{R}^3$ an hence a line in the projective plane,

[5] In $\mathcal{A}^3$, line and plane are perpendicular when they contain the right angle. The right angle is one quarter of a circle.

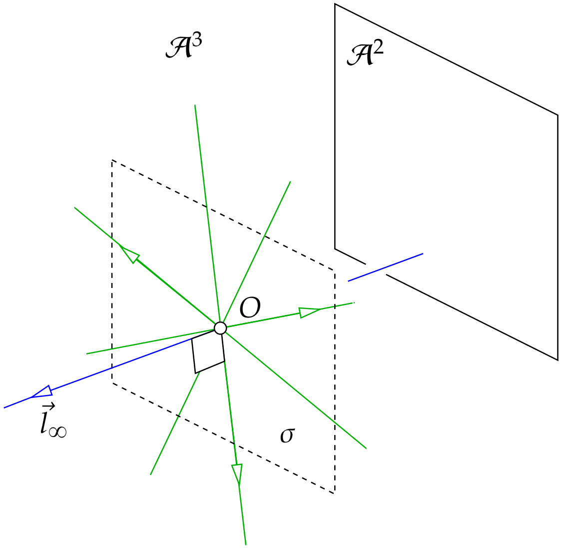


Figure 8.6: The ideal line is the set of all projective points (i.e. all lines of $A^3$ through $C$, which have no intersection with $\mathcal{A}^2$. It is a plane $\sigma$. There is exactly one, which is perpendicular to sigma, which is generated by vector $l_\infty$.

which is not in the affine plane. It is the *ideal line*[6] of the projective plane associated with the selected affine plane in that projective plane. It is represented by vector $\vec{l}_\infty$ in Figure 8.6.

For each affine plane, there is exactly one ideal line (a two-dimensional subspace of $\mathbb{R}^3$). Conversely, by selecting one line in a projective plane (i.e. one two-dimensional subspace of $\mathbb{R}^3$) the associated affine plane is de-

[6] Nevlastní přímka in Czech, line at infinity in [13].

termined as the set of all points (one-dimensional subspaces of $\mathbb{R}^3$) which are not contained in the selected ideal line (two-dimensional subspace).

### 8.2.5 Homogeneous coordinates

Once a coordinate system is fixed in an affine plane, every point of the affine plane has *unique* coordinates, which are the coordinates of its vector in the basis of the coordinate system.

A point in a real projective plane is represented by a one-dimensional subspace of $\mathbb{R}^3$. One-dimensional subspaces are represented by their bases consisting of a single non-zero vector. There are infinitely many bases representing the same one-dimensional subspace. Two basic vectors of the same one-dimensional subspace are related by a non-zero multiple.

Hence, when talking about coordinates of a point in the projective space, we actually talk about coordinates of a particular basic vector of the one-dimensional subspace that represents the point.

For instance, vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \tag{8.19}$$

are basic vectors of the same one-dimensional subspace since they are related by a non-zero multiple. These are two different “coordinates” of the same point in the real projective plane.

Hence, the “coordinates” of a point in the real projective plane are not unique. This is so radically departing from the fundamental property of coordinates, their uniqueness, that it deserves a new name. To distinguish the coordinates of a point in the affine plane, which are unique, from the “coordinates” of a point in the projective plane, which are not unique, we shall introduce new name *homogeneous coordinates*.

*Homogeneous coordinates of a point* in the real projective plane are the coordinates of a basic vector of the one-dimensional subspace, which represents the point.

*Homogeneous coordinates of a line* in the real projective plane are the coordinates of a basic vector of the one-dimensional subspace, which represents the line.

A point in an affine plane can be represented by affine as well as by homogeneous coordinates. Let us see the relationship between the two.

Let us have a point $X$ in a two-dimensional real affine plane, which is represented by coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \tag{8.20}$$

By extending the real affine plane to the real projective plane with the ideal line identified with the two-dimensional subspace $z = 0$, we can represent point $X$ by a one-dimensional subspace of $\mathbb{R}^3$ generated by its basic vector

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{8.21}$$

Thus, $X$ has affine coordinates $\begin{bmatrix} x & y \end{bmatrix}^\top$ and homogeneous coordinates $\begin{bmatrix} u & v & w \end{bmatrix}^\top$, where $u = \lambda\, x$, $v = \lambda\, y$, and $w = \lambda\, 1$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Ideal points do not have affine coordinates. Their homogeneous coordinates are

$$\begin{bmatrix} x & y & 0 \end{bmatrix}^\top \tag{8.22}$$

where $x, y \in \mathbb{R}$ and either $x \neq 0$ or $y \neq 0$.

The zero vector $\vec{0}$ is not a basis of any one-dimensional space and thus represents neither a point nor a line.

### 8.2.6 Incidence of points and lines

We say that a point $x$ is incident with line $l$ if and only if it can generate the line with another point $y$, Figure 8.7. In the representation of subspaces

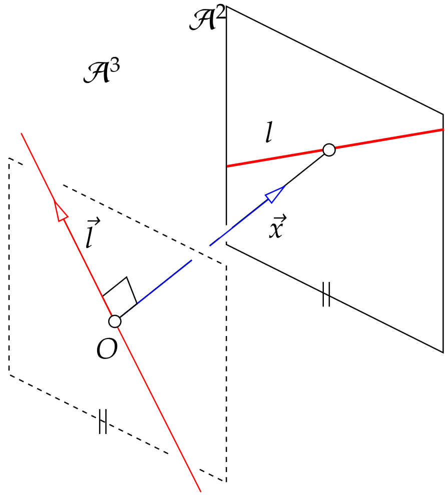


Figure 8.7: A point $x$ is incident with a line $l$ if and only if it can generate the line with another point $y$. Lines in $\mathcal{A}^3$ representing the point and the line are perpendicular to each other.

of $\mathbb{R}^3$, it means that

$$\vec{l}_{\bar{\beta}}^{\top} \vec{x}_{\beta} = 0 \tag{8.23}$$

This means that the one-dimensional subspace of $\mathbb{R}^3$ representing the line is orthogonal to the one-dimensional subspace of $\mathbb{R}^3$ representing the point w.r.t. the standard (Euclidean) scalar product. In the geometrical model of the real projective plane it means that the line of $\mathcal{A}^3$ representing $x$ is perpendicular to line of $\mathcal{A}^3$ representing $l$.

Let us write explicitly the coordinates of the bases generating the one-dimensional subspaces as

$$\vec{x}_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{l}_{\bar{\beta}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

then we get

$$a\,x + b\,y + c\,z = 0$$

and for affine points represented with $z = 1$ this formula reduces to

$$a\,x + b\,y + c = 0$$

which is the familiar equation of a line in the two dimensional real affine plane.

### 8.2.7 Join of points

Every two distinct points $x$ and $y$ in the real projective plane are incident with exactly one line $l$. The *join* of two distinct points is the unique line passing through them.

In the real projective plane, two distinct points are represented by two different one-dimensional subspaces with bases $\vec{x}$ and $\vec{y}$.

The homogeneous coordinates of this line, i.e. the coordinates of the basic vectors of the one-dimensional subspace representing the line, can be obtained by solving the following system of homogeneous equations for coordinates of the vector $\vec{l}$

$$\vec{l}_{\bar{\beta}}^{\top} \vec{x}_\beta = 0 \tag{8.24}$$

$$\vec{l}_{\bar{\beta}}^{\top} \vec{y}_\beta = 0 \tag{8.25}$$

w.r.t. $\beta$ and $\bar{\beta}$ in $\mathbb{R}^3$. The set of solutions forms the one-dimensional subspace that represents the line $l$.

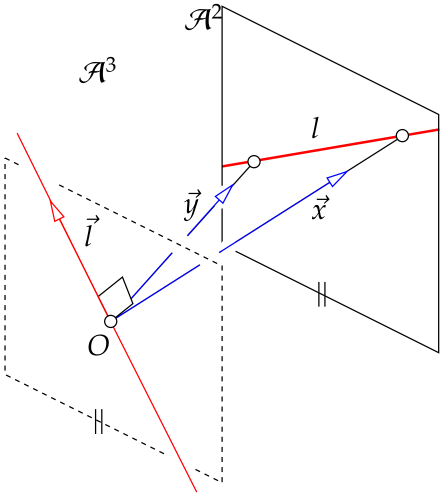


Figure 8.8: The join of two distinct points is the unique line passing through them.

We have seen in Section 1.3 that vector $\vec{l}_{\vec{\beta}}$ can be conveniently constructed by the vector product as

$$\vec{l}_{\vec{\beta}} = \vec{x}_\beta \times \vec{y}_\beta \tag{8.26}$$

Notice, that *in the real projective plane as well as in the real affine plane, there is exactly one line incident with two distinct points*.

### 8.2.8 Meet of lines

Every two distinct lines $k$ and $l$ in a projective plane are incident exactly to one point $x$. The *meet* of two distinct lines is the unique point incident with them.

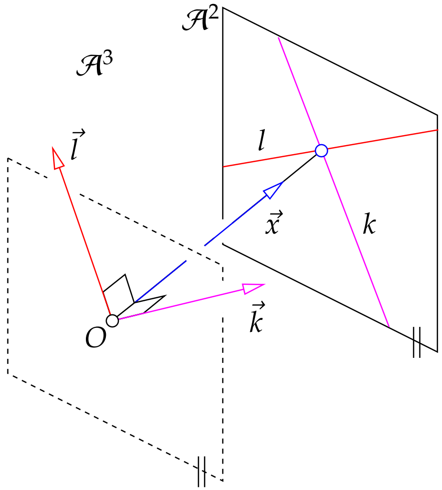


Figure 8.9: The meet of two distinct lines is the unique point incident with them.

In the real projective plane, two distinct lines are represented by two different one-dimensional subspaces with bases $\vec{k}$ and $\vec{l}$.

The homogeneous coordinates of this point, i.e. the coordinates of the vectors in the one-dimensional subspace representing the point, can be obtained by solving the following system of homogeneous equations for coordinates of the vector $\vec{x}$ w.r.t. $\beta$ in $\mathbb{R}^3$

$$\begin{aligned} \vec{k}_{\bar{\beta}}^{\top} \vec{x}_{\beta} &= 0 \\ \vec{l}_{\bar{\beta}}^{\top} \vec{x}_{\beta} &= 0 \end{aligned}$$

The set of solutions forms the one-dimensional subspace that represents point $x$. To get one basic vector in the subspace, we may again employ the vector product in $\mathbb{R}^3$ and compute

$$\vec{x}_\beta = \vec{k}_{\bar{\beta}} \times \vec{l}_{\bar{\beta}}$$

Notice, that *in the real projective plane there is exactly one point incident to two distinct lines*.

This is not true in an affine plane because there are (parallel) lines that have no point in common!

## 8.3 Line coordinates under homography

Let us now investigate the behavior of homogeneous coordinates of lines in projective plane mapped by a homography.

Let us have two points represented by vectors $\vec{x}_\beta$, $\vec{y}_\beta$. We now map the points, represented by vectors $\vec{x}_\beta$, $\vec{y}_\beta$, by a homography, represented by matrix $\mathtt{H}$, to points represented by vectors $\vec{x}'_{\beta'}$, $\vec{y}'_{\beta'}$ such that there are $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1\lambda_2 \neq 0$

$$\lambda_1\, \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta \tag{8.27}$$

$$\lambda_2\, \vec{y}'_{\beta'} = \mathtt{H}\, \vec{y}_\beta \tag{8.28}$$

Homogeneous coordinates $\vec{p}_{\bar{\beta}}$ of the line passing through points represented by $\vec{x}_\beta$, $\vec{y}_{\bar{\beta}}$ and homogeneous coordinates $\vec{p}'_{\bar{\beta}'}$ of the line passing through points represented by $\vec{x}'_{\beta'}$, $\vec{y}'_{\beta'}$ are obtained by solving the linear systems

$$\vec{p}_{\bar{\beta}}^\top\, \vec{x}_\beta = 0 \quad \text{and} \quad \vec{p}'^\top_{\bar{\beta}'}\, \vec{x}'_{\beta'} = 0 \tag{8.29}$$

$$\vec{p}_{\bar{\beta}}^\top\, \vec{y}_\beta = 0 \quad \quad \vec{p}'^\top_{\bar{\beta}'}\, \vec{y}'_{\beta'} = 0 \tag{8.30}$$

for a non-trivial solutions. Writing down the incidence of points and lines, we get

$$\begin{aligned}
\lambda_1 \vec{p}_{\bar\beta}^\top \mathtt{H}^{-1} \vec{x}'_{\beta'} = 0 \quad &\Leftrightarrow \quad \vec{p}_{\bar\beta}^\top \mathtt{H}^{-1} \vec{x}'_{\beta'} = 0 \\
\lambda_2 \vec{p}_{\bar\beta}^\top \mathtt{H}^{-1} \vec{y}'_{\beta'} = 0 \quad &\Leftrightarrow \quad \vec{p}_{\bar\beta}^\top \mathtt{H}^{-1} \vec{y}'_{\beta'} = 0
\end{aligned}$$

We see that $\vec{p}'_{\bar\beta'}$ and $\mathtt{H}^{-\top}\vec{p}_{\bar\beta}$ are solutions of the same set of homogeneous equations. When $\vec{x}_\beta$, $\vec{y}_\beta$ are independent, then there is $\lambda \in \mathbb{R}$ such that

$$\lambda\, \vec{p}'_{\bar\beta'} = \mathtt{H}^{-\top}\vec{p}_{\bar\beta} \tag{8.31}$$

since the solution space is one-dimensional. Equation 8.31 gives the relationship between homogeneous coordinates of a line and its image under homography $\mathtt{H}$.

### 8.3.1 Join under homography

Let us go one step further and establish formulas connecting line coordinates constructed by vector products. Construct joins as

$$\vec{p}_{\bar\beta} = \vec{x}_\beta \times \vec{y}_\beta \quad \text{and} \quad \vec{p}'_{\bar\beta'} = \vec{x}'_{\beta'} \times \vec{y}'_{\beta'} \tag{8.32}$$

and use Equation 1.50 to get

$$\vec{x}_{\beta'} \times \vec{y}_{\beta'} = \frac{\mathtt{H}^{-\top}}{|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta) \tag{8.33}$$

$$(\lambda_1\, \vec{x}'_{\beta'}) \times (\lambda_2\, \vec{y}'_{\beta'}) = \frac{\mathtt{H}^{-\top}}{|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta) \tag{8.34}$$

$$\vec{x}'_{\beta'} \times \vec{y}'_{\beta'} = \frac{\mathtt{H}^{-\top}}{\lambda_1\, \lambda_2\, |\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta) \tag{8.35}$$

$$\vec{p}'_{\bar\beta'} = \frac{\mathtt{H}^{-\top}}{\lambda_1\, \lambda_2\, |\mathtt{H}^{-\top}|}\vec{p}_{\bar\beta} \tag{8.36}$$

### 8.3.2 Meet under homography

Let us next look at the meet. Let point $\vec{x}$ be the meet of lines $\vec{p}$ and $\vec{q}$ with line cordinates $\vec{p}_{\bar{\beta}}$, $\vec{q}_{\bar{\beta}}$, which are related by a homography $\mathtt{H}$ to line coordinates $\vec{p}'_{\bar{\beta}'}$ and $\vec{q}'_{\bar{\beta}'}$ by

$$\lambda_1 \vec{p}'_{\bar{\beta}'} = \mathtt{H}^{-\top} \vec{p}_{\bar{\beta}} \tag{8.37}$$
$$\lambda_2 \vec{q}'_{\bar{\beta}'} = \mathtt{H}^{-\top} \vec{q}_{\bar{\beta}} \tag{8.38}$$

for some non-zero $\lambda_1$, $\lambda_2$. Construct meets as

$$\vec{x}_\beta = \vec{p}_{\bar{\beta}} \times \vec{q}_{\bar{\beta}} \quad \text{and} \quad \vec{x}'_{\beta'} = \vec{p}'_{\bar{\beta}'} \times \vec{q}'_{\bar{\beta}'} \tag{8.39}$$

and, similarly as above, use Equation 1.50 to get

$$\vec{x}'_{\beta'} = \frac{(\mathtt{H}^{-\top})^{-\top}}{\lambda_1 \lambda_2 \, |(\mathtt{H}^{-\top})^{-\top}|} \vec{x}_\beta = \frac{\mathtt{H}}{\lambda_1 \lambda_2 \, |\mathtt{H}|} \vec{x}_\beta \tag{8.40}$$

### 8.3.3 Meet of join under homography

We can put the above together to get meet of join under homography. We consider two pairs of points represented by their homogeneous coordinates $\vec{x}_\beta$, $\vec{y}_\beta$, and $\vec{z}_\beta$, $\vec{w}_\beta$ and the corresponding pairs of points with their homogeneous coordinates $\vec{x}'_{\beta'}$, $\vec{y}'_{\beta'}$, and $\vec{z}'_{\beta'}$, $\vec{w}'_{\beta'}$ related by homography $\mathtt{H}$ as

$$\lambda_1 \vec{x}'_{\beta'} = \mathtt{H}\vec{x}_\beta, \quad \lambda_2 \vec{y}'_{\beta'} = \mathtt{H}\vec{y}_\beta, \quad \lambda_3 \vec{z}'_{\beta'} = \mathtt{H}\vec{z}_\beta, \quad \lambda_4 \vec{w}'_{\beta'} = \mathtt{H}\vec{w}_\beta \tag{8.41}$$

Let us now consider point

$$\vec{v}'_{\beta'} = (\vec{x}'_{\beta'} \times \vec{y}'_{\beta'}) \times (\vec{z}'_{\beta'} \times \vec{w}'_{\beta'}) \tag{8.42}$$

$$= \left(\frac{\mathtt{H}^{-\top}}{\lambda_1\,\lambda_2\;|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta)\right) \times \left(\frac{\mathtt{H}^{-\top}}{\lambda_3\,\lambda_4\;|\mathtt{H}^{-\top}|}(\vec{z}_\beta \times \vec{w}_\beta)\right) \tag{8.43}$$

$$= \frac{\mathtt{H}\,|\mathtt{H}|}{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4}(\vec{x}_\beta \times \vec{y}_\beta) \times (\vec{z}_\beta \times \vec{w}_\beta) \tag{8.44}$$

$$= \frac{\mathtt{H}\,|\mathtt{H}|}{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4}\vec{v}_\beta \tag{8.45}$$

### 8.3.4 Note on homographies that are rotations

First notice that homogeneous coordinats of points and lines constructed as combinations of joins and meets indeed behave under a homography as homogeneous coordinates constructed from affine coordinates of points.

Secondly, when the homography is a rotation and homogeneous coordinates are unit vecors, all $\lambda$'s become equal to one, the determinant of $\mathtt{H}$ is one and $\mathtt{H}^{-\top} = \mathtt{H}$. Therefore, all homogeneous coordinates in the previous formulas become related just by $\mathtt{H}$.

## 8.4 Vanishing points

When modeling perspective projection in the affine space with affine projection planes, we meet somewhat unpleasant situations. For instance, imagine a projection of two parallel lines $K, L$, which are in a plane $\tau$ in the space into the projection plane $\pi$ through the center $C$, Figure 8.10.

The lines $K, L$ project to image lines $k, l$. As we go with two points $X, Y$ along the lines $k, l$ away from the projection plane, their images $x, y$ get closer and closer to the point $v$ in the image but they do not reach point $v$. We shall call this point of convergence of lines $K$, $L$ the *vanishing point*[7].

[7] Úběžník in Czech.

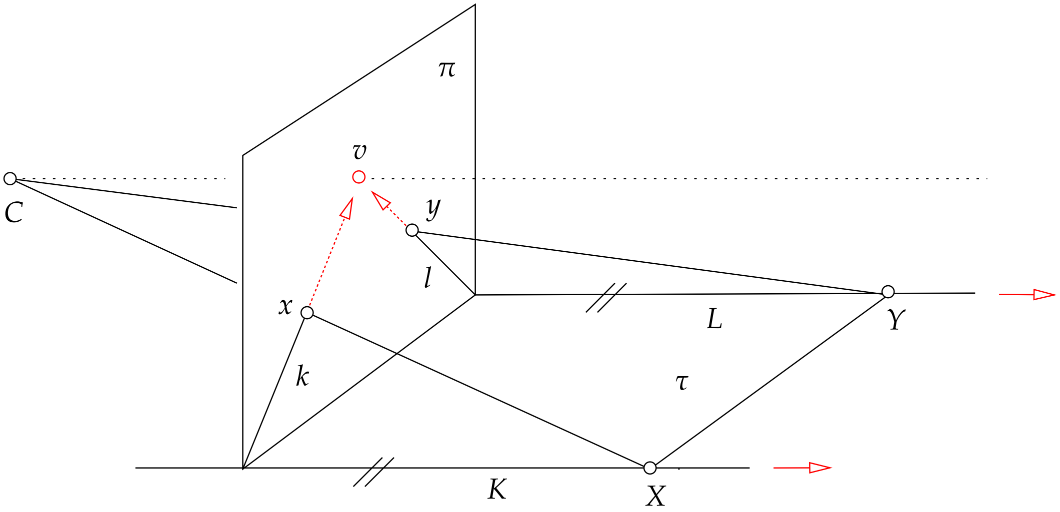


Figure 8.10: Vanishing point $v$ is the point towards projections $x$ an $y$ tend as $X$ and $Y$ move away from $\pi$ but which they never reach.

## 8.5 Vanishing line and horizon

If we take all sets of parallel lines in $\tau$, each set with a different direction, then all the points of convergence in the image will fill a complete line $h$.

The line $h$ is called the *vanishing line* or the *horizon*[8] when $\tau$ is the ground plane.

Now, imagine that we project all points from $\tau$ to $\pi$ using the affine geometrical projection model. Then, no point from $\tau$ will project to $h$. Similarly, when projecting in the opposite direction, i.e. $\pi$ to $\tau$, line $h$ has no image, i.e. it does not project anywhere to $\tau$.

[8]Horizont in Czech

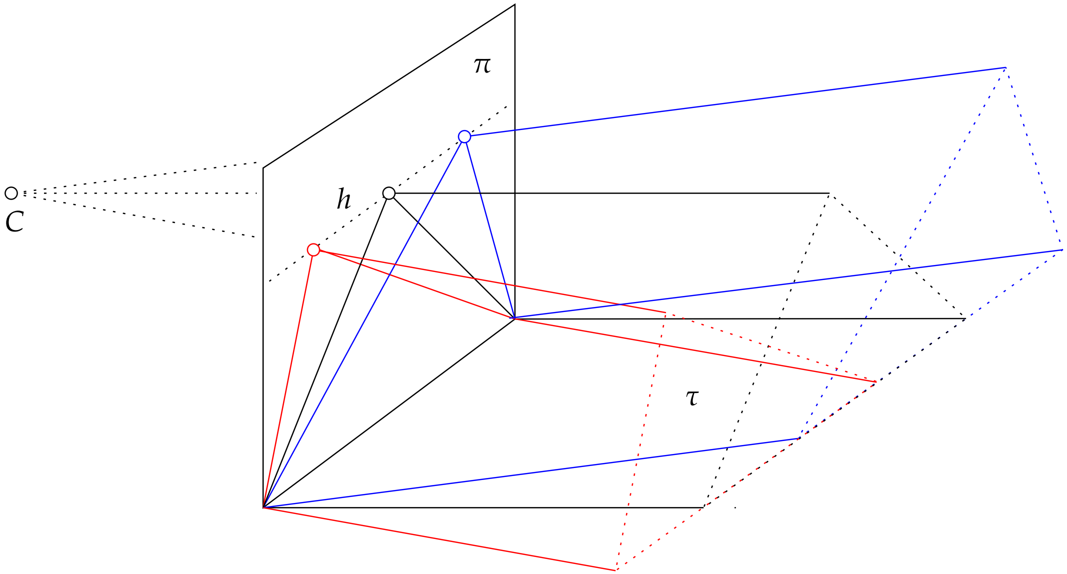


Figure 8.11: Vanishing line (horizon) $h$ is the line of vanishing points.

When using the affine geometrical projection model with the real projective plane to model the perspective projection (which is equivalent to the algebraic model in $\mathbb{R}^3$), all points of the projective plane $\tau$ (obtained as the projective completion of the affine plane $\tau$) will have exactly one image in the projective plane $\pi$ (obtained as the projective completion of the affine plane $\pi$) and vice versa. This total symmetry is useful and beautiful.