Digital Image

(B4M33DZO, Summer 2024)

Lecture 8:

Image Registration 2

https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start

Daniel Sýkora & Ondřej Drbohlav

Department of Cybernetics

Faculty of Electrical Engineering

Czech Technical University in Prague

Translation

Minimize similarity metric $\circ$ over all possible shifts:

$$\arg\min_{[s,t]} \sum_x \sum_y \mathbf{A}[x+s, y+t] \circ \mathbf{B}[x, y]$$

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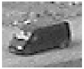

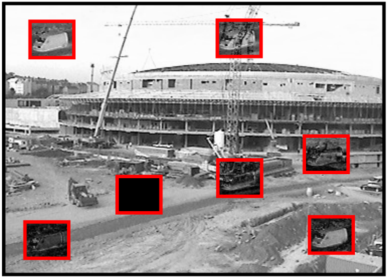

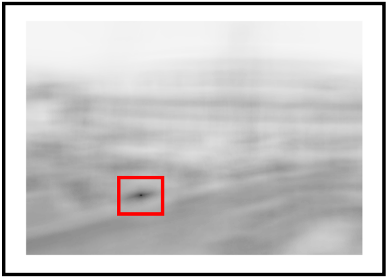

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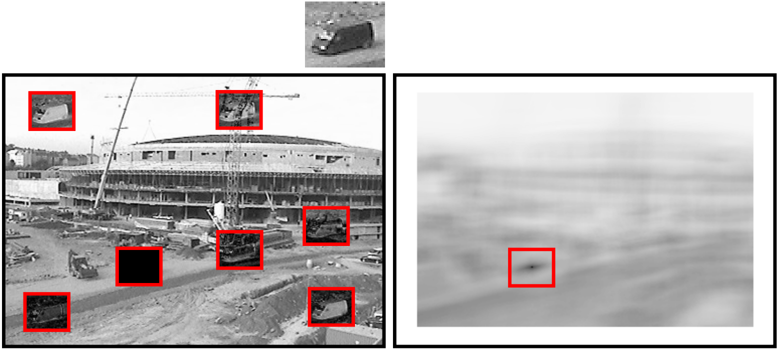

Recall: complexity of block matching is $\mathcal{O}(|\mathbf{A}| \cdot |\mathbf{B}|)$.

Translation (speed-ups)

1. **Early termination** ($\mathcal{O}(|\mathbf{A}| \cdot |\mathbf{B}|)$, global minimum):
   Compare current summation with the last best value.

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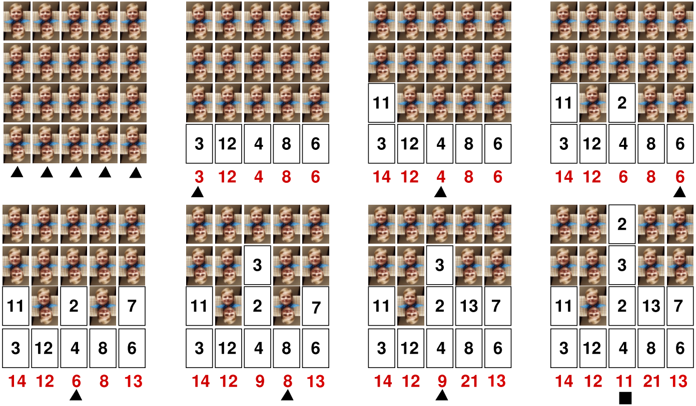
2. **Winner-update strategy** ($\mathcal{O}(|\mathbf{A}| \cdot |\mathbf{B}|)$, global minimum):



Translation (speed-ups)

3. Hierarchical approach ($\mathcal{O}(|\mathbf{A}| \cdot \log |\mathbf{A}|)$, local minimum):

Use reduced resolution to compute initial solution and refine.

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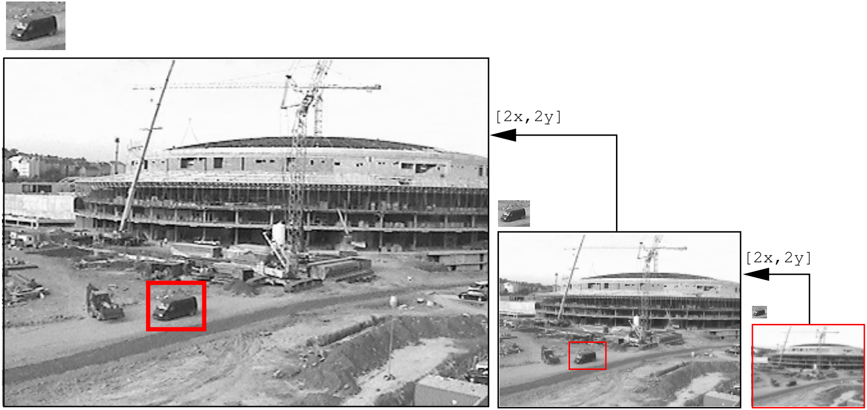
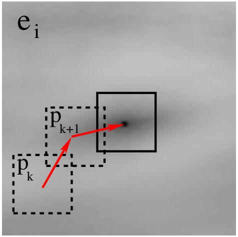


Translation (speed-ups)

4. Gradient descent ($\mathcal{O}(|\mathbf{B}|)$, local minimum):



We want to minimize:

$$E = \sum_i \left(\mathbf{A}[\mathbf{x}_i + \mathbf{t}] - \mathbf{B}[\mathbf{x}_i]\right)^2 \quad \Rightarrow \quad E' = 0$$

Translation (speed-ups)

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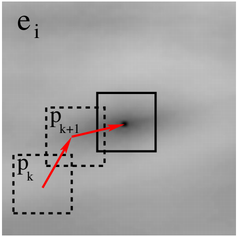

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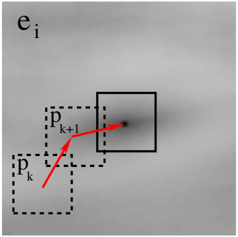

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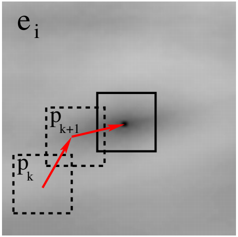

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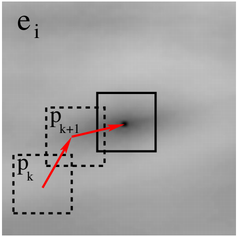

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$$\mathbf{t} = \left(\sum_i \left(\mathbf{A}'_i\right)^T \left(\mathbf{A}'_i\right)\right)^{-1} \left(\sum_i \left(\mathbf{A}'_i\right)^T \left(\mathbf{B}_i - \mathbf{A}_i\right)\right)$$

Affine & Projective

Generalized gradient descent:

$$E = \sum_i \left(\mathbf{A}[\mathbf{W}(\mathbf{x}_i, \mathbf{p})] - \mathbf{B}[\mathbf{x}_i]\right)^2 \approx \sum_i \left(\mathbf{A}_i + \mathbf{A}'_i \mathbf{J} \mathbf{p} - \mathbf{B}_i\right)^2$$

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$$E' = 2 \sum_i \left(\mathbf{A}'_i \mathbf{J}\right)^T \left(\mathbf{A}_i + \mathbf{A}'_i \mathbf{J}\mathbf{p} - \mathbf{B}_i\right) = 0$$

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$$\mathbf{p} = \mathbf{H}^{-1} \sum_i \left(\mathbf{A}_i'\mathbf{J}\right)^T \left(\mathbf{B}_i - \mathbf{A}_i\right)$$

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$$\mathbf{J} = \underbrace{\begin{pmatrix} \frac{\partial \mathbf{W}_\mathbf{x}}{\partial \mathbf{p}_1} & \frac{\partial \mathbf{W}_\mathbf{x}}{\partial \mathbf{p}_2} & \cdots & \frac{\partial \mathbf{W}_\mathbf{x}}{\partial \mathbf{p}_n} \\ \frac{\partial \mathbf{W}_\mathbf{y}}{\partial \mathbf{p}_1} & \frac{\partial \mathbf{W}_\mathbf{y}}{\partial \mathbf{p}_2} & \cdots & \frac{\partial \mathbf{W}_\mathbf{y}}{\partial \mathbf{p}_n} \end{pmatrix}}_{\text{Jacobian}} \qquad \mathbf{H} = \sum_i \left(\mathbf{A}'_i \mathbf{J}\right)^T \left(\mathbf{A}'_i \mathbf{J}\right)$$

Jacobian

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Affine transformation: $\mathbf{p} = (a_{11}, a_{12}, a_{21}, a_{22}, x_0, y_0)$

$$x' = a_{11}x + a_{12}y + x_0 \qquad y' = a_{21}x + a_{22}y + y_0$$

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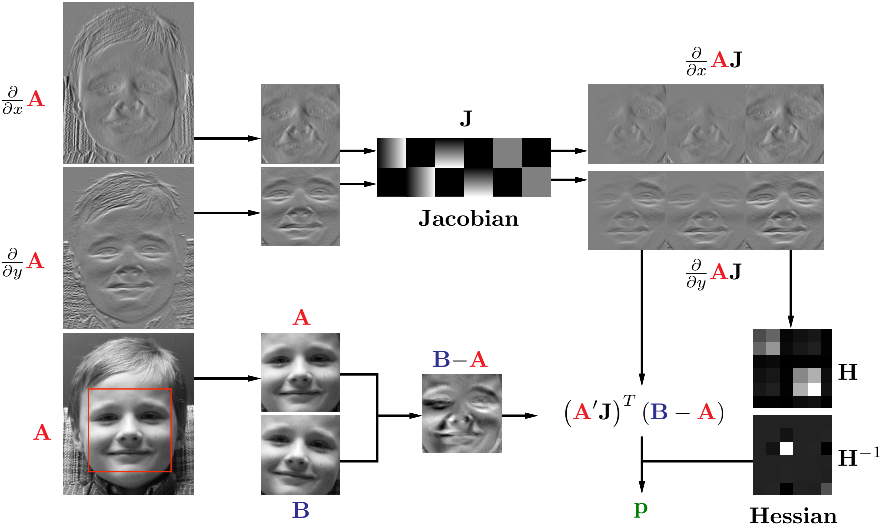
$$\mathbf{J} = \frac{1}{f_1 x + f_2 y + 1} \begin{pmatrix} x & y & 0 & 0 & 1 & 0 & -xx' & -yx' \\ 0 & 0 & x & y & 0 & 1 & -xy' & -yy' \end{pmatrix}$$

Affine


$\frac{\partial}{\partial x}\mathbf{A}$
$\frac{\partial}{\partial y}\mathbf{A}$
$\mathbf{A}$
$\mathbf{J}$
Jacobian
$\frac{\partial}{\partial x}\mathbf{A}\mathbf{J}$
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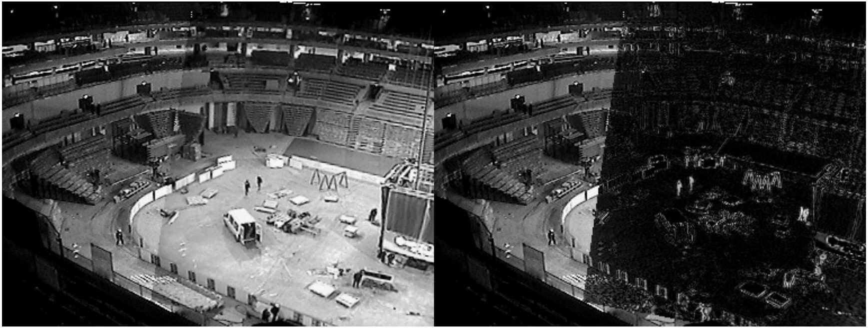
$\mathbf{B}$
$\mathbf{B}-\mathbf{A}$
$\left(\mathbf{A}'\mathbf{J}\right)^T(\mathbf{B}-\mathbf{A})$
$\mathbf{H}$
$\mathbf{H}^{-1}$
Hessian
$\mathbf{p}$

Hierarchical approach

Hierarchical gradient descent

1. **Model: translation ⇒ rigid ⇒ affine ⇒ projective**
2. **Scale: 32x32 ⇒ 64x64 ⇒ 128x128 ⇒ 256x256**

Composite **Error**



Interest Points

Good localization:

$$\sum_x \sum_y \left(\mathbf{I}[x,y] - \mathbf{I}[x+s, y+t]\right)^2 > 0 \qquad \forall (s,t) : ||(s,t)|| > 0$$

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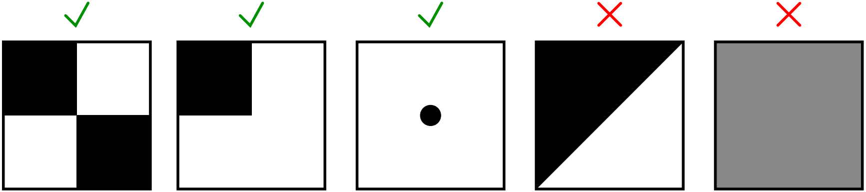

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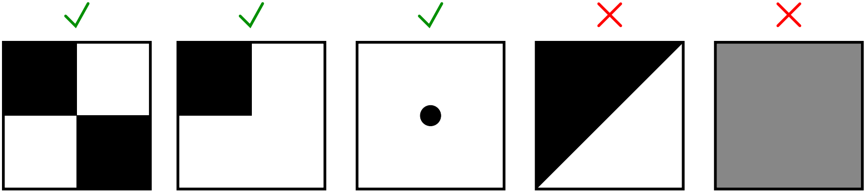
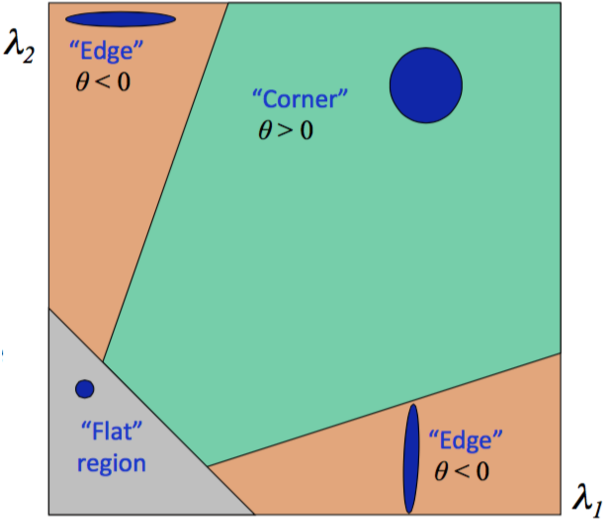


Harris detector:

$$\mathbf{M} = \begin{pmatrix} \sum \left(\frac{\partial \mathbf{I}}{\partial x}\right)^2 & \sum \frac{\partial \mathbf{I}}{\partial x}\frac{\partial \mathbf{I}}{\partial y} \\ \sum \frac{\partial \mathbf{I}}{\partial x}\frac{\partial \mathbf{I}}{\partial y} & \sum \left(\frac{\partial \mathbf{I}}{\partial y}\right)^2 \end{pmatrix} \qquad R = \det(\mathbf{M}) - k\left(\mathrm{trace}(\mathbf{M})\right)^2$$

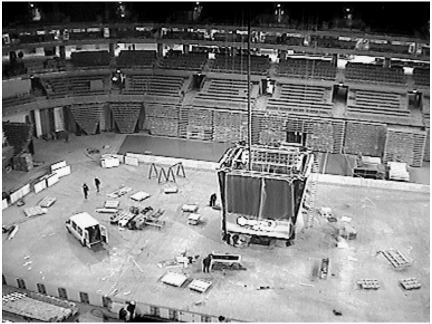
Interest Points



Interest Points

Illustration: Repeat until specified number of points was found:

1. **find pixel (x, y) where $R[x, y]$ has maximal value**
2. **compute centroid (c_x, c_y) in a neighborhood of (x, y)**
3. **store the centroid to the list of interesting points**
4. **remove point by drawing zero circle around (x, y) in R**



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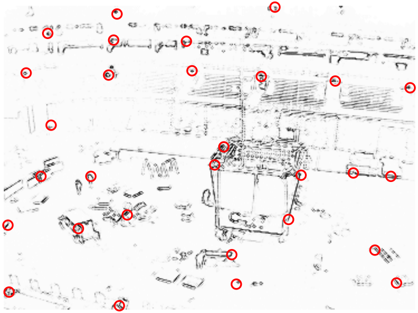
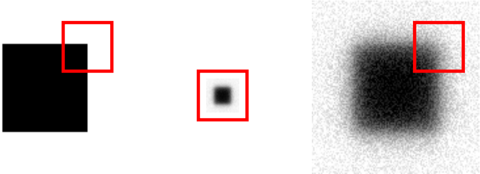


Interest Points

Problem: output of Harris detector depends on the selected scale.



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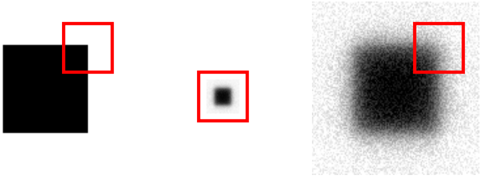

Invariance: e.g. SIFT (Scale-invariant feature transform):

Local extremes of normalized Laplacian-of-Gaussian scale-space.

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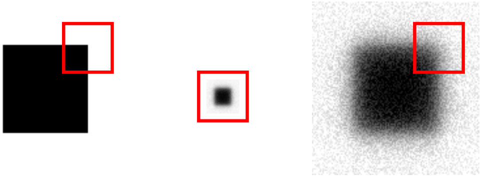

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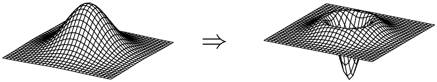
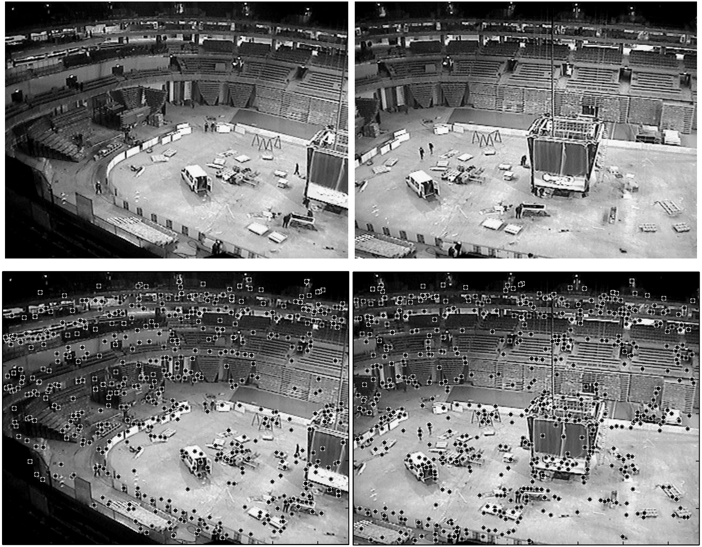
Local extremes of normalized Laplacian-of-Gaussian scale-space.

$$\mathbf{L} \circ \mathbf{G} = \nabla^2 \left(\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \right) = \frac{1}{\pi\sigma^4} \left(\frac{x^2+y^2}{2\sigma^2} - 1 \right) e^{-\frac{x^2+y^2}{2\sigma^2}}$$

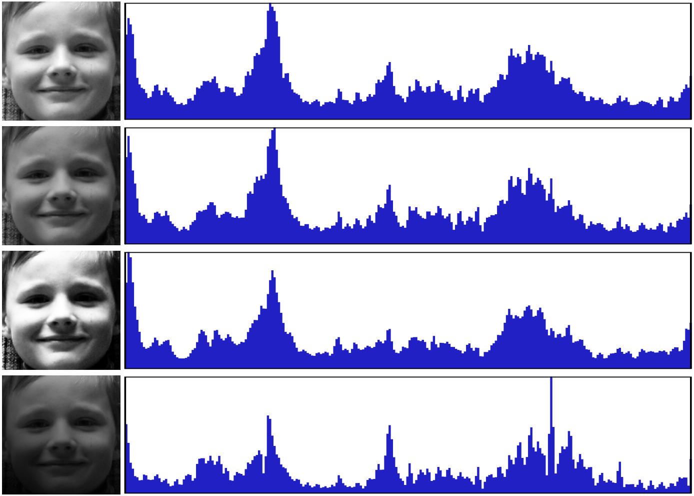


Interest Points



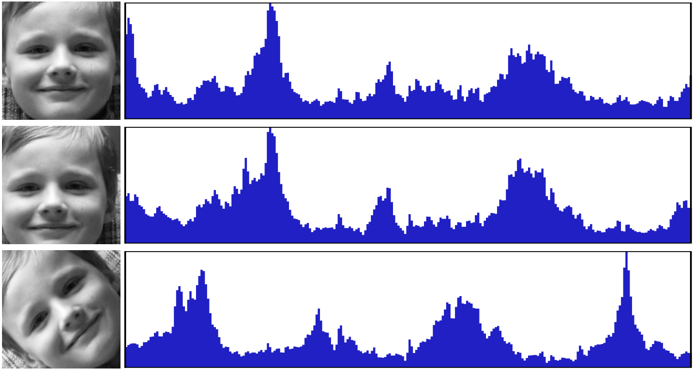
Point descriptor

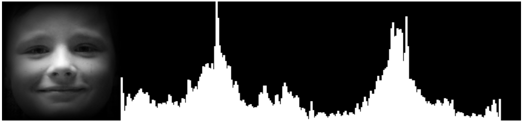
Histogram of gradient orientations (illumination changes):



Point descriptor

Histogram of gradient orientations (rotation & translation):





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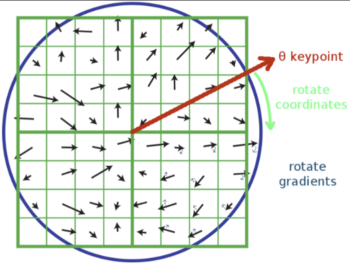

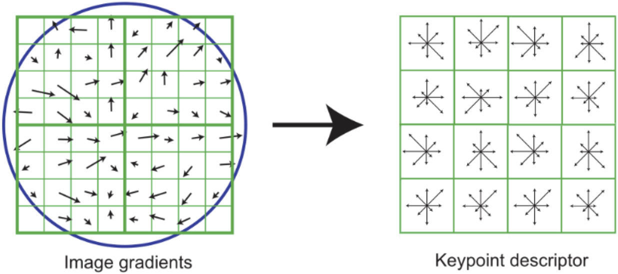

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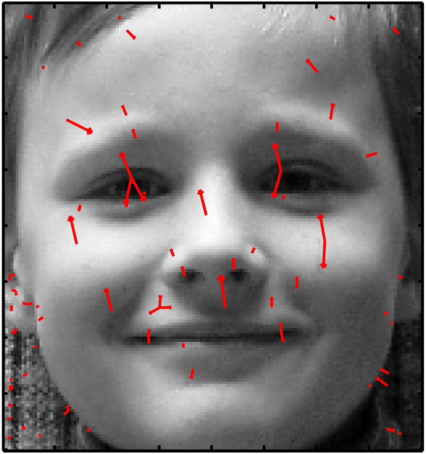

1. **Besides location also orientation and scale are important.**
2. **Single location may contain multiple interesting points.**

Estimating model parameters

Select $\geq M$ best matching point correspondences:

$$x' = \frac{a_{11}x + a_{12}y + x_0}{f_1 x + f_2 y + f_3} \qquad y' = \frac{a_{21}x + a_{22}y + y_0}{f_1 x + f_2 y + f_3}$$

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Select $\geq M$ best matching point correspondences:

$$x' = \frac{a_{11}x + a_{12}y + x_0}{f_1 x + f_2 y + f_3} \qquad y' = \frac{a_{21}x + a_{22}y + y_0}{f_1 x + f_2 y + f_3}$$

Build (overdetermined) linear system (solvable by SVD):

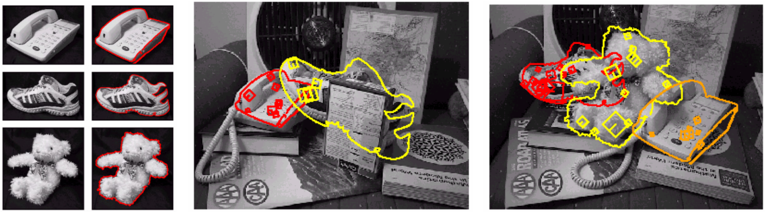
$$\begin{pmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x'_1 x_1 & -x'_1 y_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1 x_1 & -y'_1 y_1 & -y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x'_2 x_2 & -x'_2 y_2 & -x'^2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -y'_2 x_2 & -y'_2 y_2 & -y'_2 \\ . & . & . & . & . & . & . & . & \\ . & . & . & . & . & . & . & . & \\ x_M & y_M & 1 & 0 & 0 & 0 & -x'_M x_M & -x'_M y_M & -x'_M \\ 0 & 0 & 0 & x_M & y_M & 1 & -y'_M x_M & -y'_M y_M & -y'_M \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{12} \\ x_0 \\ a_{21} \\ a_{22} \\ y_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{pmatrix}$$

Applications

Hyper-resolution panorama stitching:

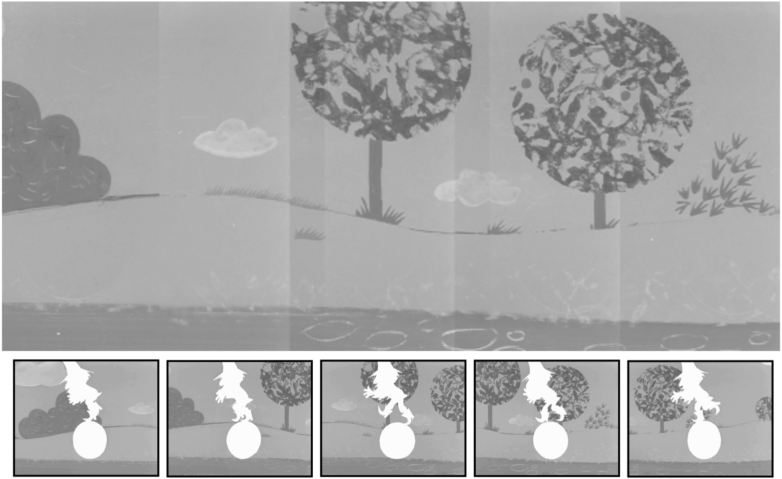
Applications

Image and object retrieval:



Applications

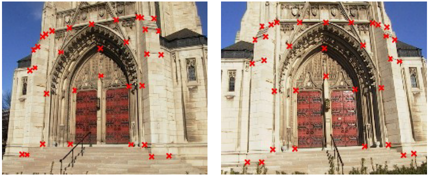
Recovering background from occluded observations:



Applications

Structure from motion:

source images



novel view synthesis