

Digital Image

(B4M33DZO, Summer 2024)

Lecture 7:

Image Deformation

<https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start>

Daniel Sýkora & Ondřej Drbohlav

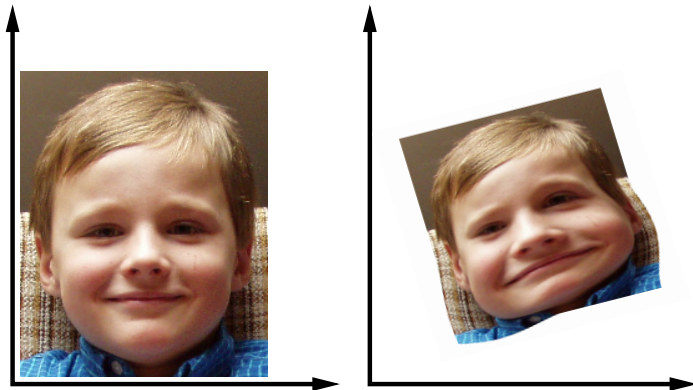
Department of Cybernetics

Faculty of Electrical Engineering

Czech Technical University in Prague

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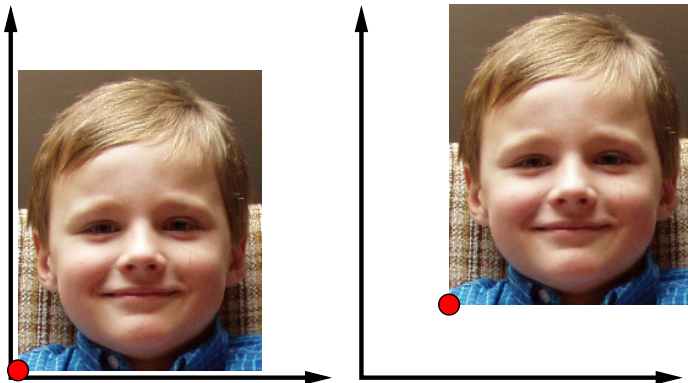




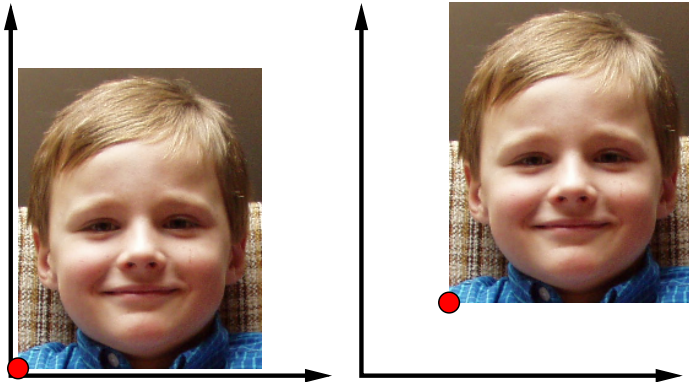
1 point

$$x' = x + x_0$$

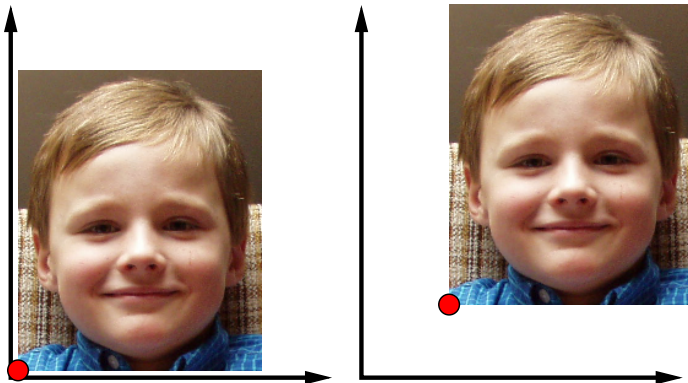
$$y' = y + y_0$$



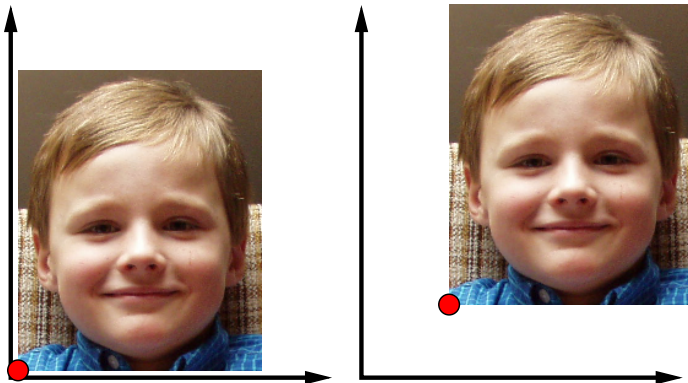
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



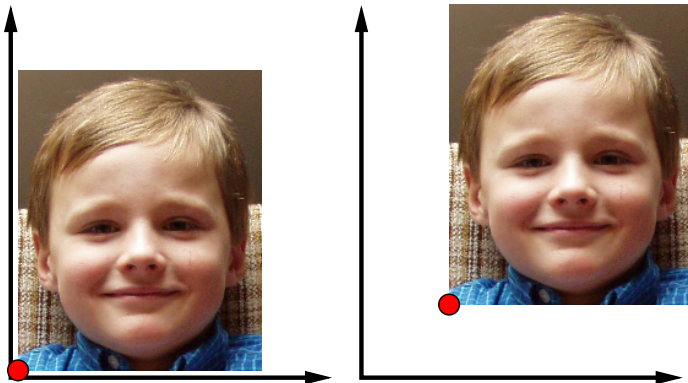
$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



$$\mathbf{x}' = \mathbf{T}\mathbf{x}$$



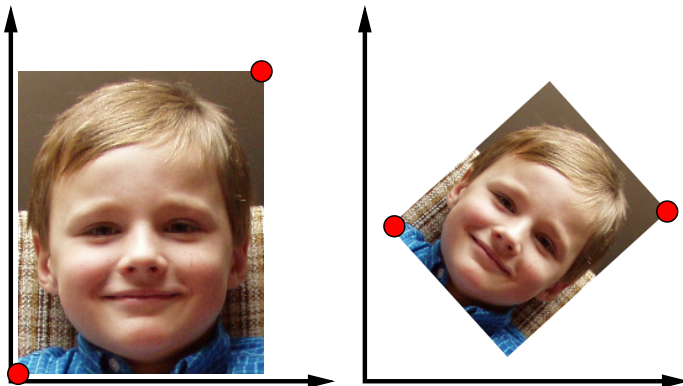
$$\mathbf{x}' = \mathbf{T}\mathbf{x}, \quad \mathbf{x} = \mathbf{T}^{-1}\mathbf{x}'$$



2-point:
(translation)

$$x' = x + x_0$$

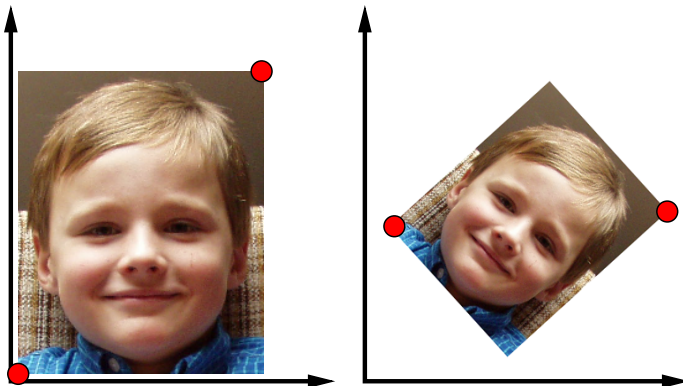
$$y' = y + y_0$$



2-point:
(translation + scale)

$$x' = \mu x + x_0$$

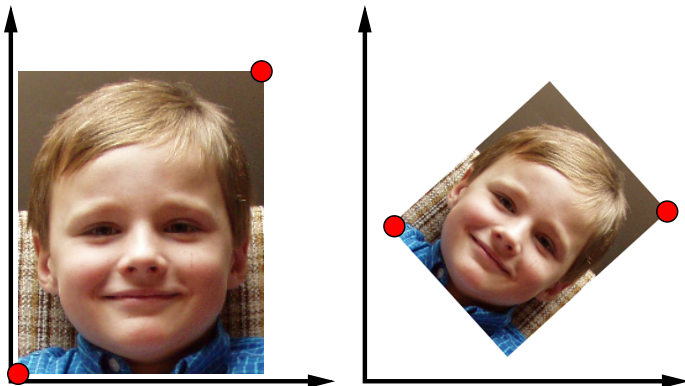
$$y' = \mu y + y_0$$



2-point:
(translation + scale + rotation)

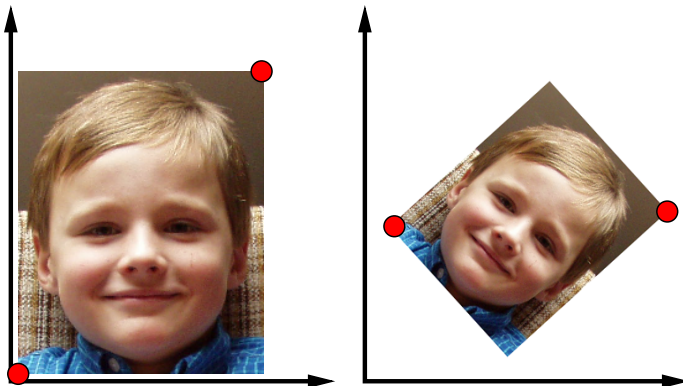
$$x' = \mu \cos(\alpha)x + \mu \sin(\alpha)y + x_0$$

$$y' = -\mu \sin(\alpha)x + \mu \cos(\alpha)y + y_0$$



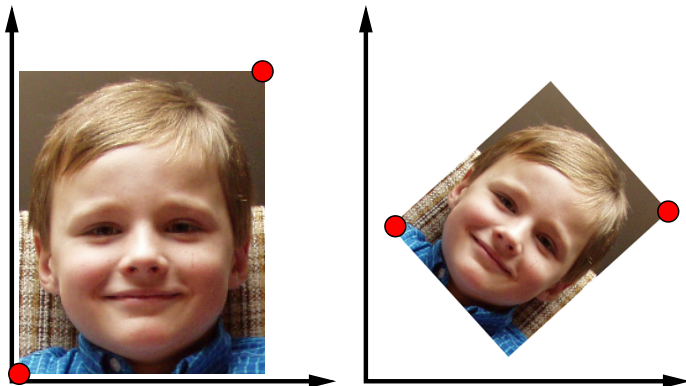
2-point:
(translation + scale + rotation)

$$\mathbf{x}' = \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{R} \cdot \mathbf{x}$$



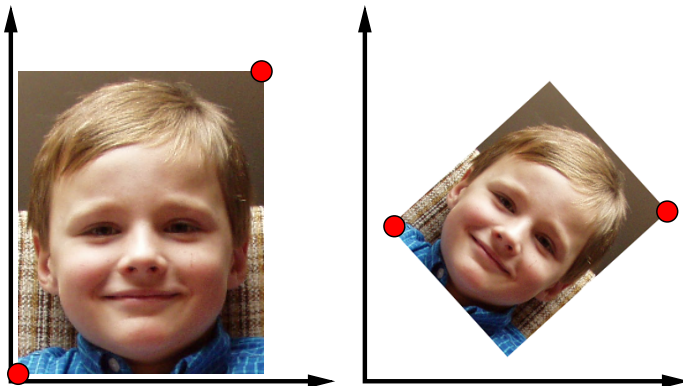
2-point:
(translation)

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



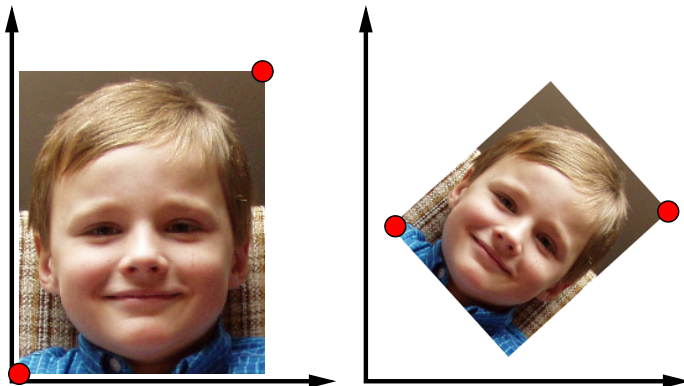
2-point:
(scale)

$$M = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



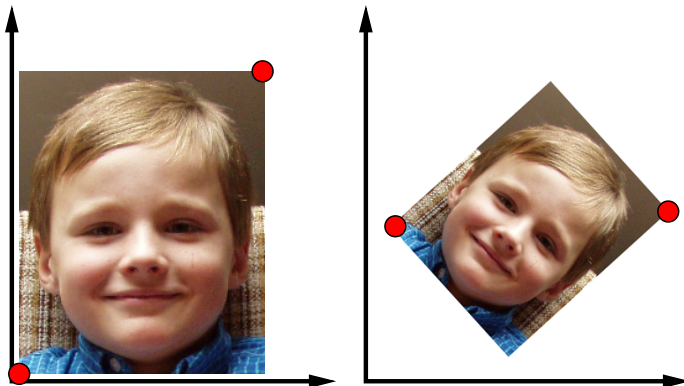
**2-point:
(rotation)**

$$\mathbf{R} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



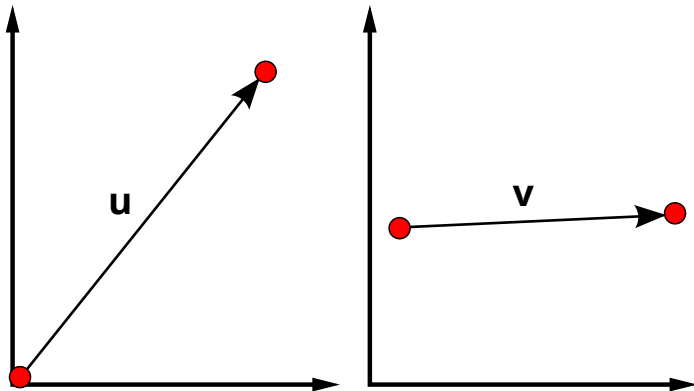
2-point:
(translation + scale + rotation)

$$S = \begin{pmatrix} \mu \cos(\alpha) & \mu \sin(\alpha) & x_0 \\ -\mu \sin(\alpha) & \mu \cos(\alpha) & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

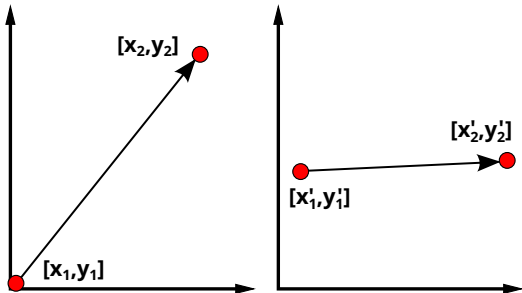


2-point:
(scale + rotation)

$$\mu = \frac{|\mathbf{v}|}{|\mathbf{u}|} \quad \cos(\alpha) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}$$



2-point: $x' = ax + by + x_0$
(translation + scale + rotation) $y' = -bx + ay + y_0$



solve a linear system:

$$x'_1 = ax_1 + by_1 + x_0$$

$$y'_1 = -bx_1 + ay_1 + y_0$$

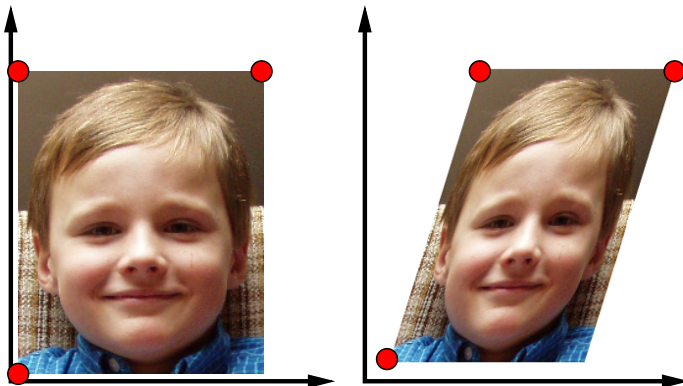
$$x'_2 = ax_2 + by_2 + x_0$$

$$y'_2 = -bx_2 + ay_2 + y_0$$

3-point:
(translation + scale + rotation + skew)

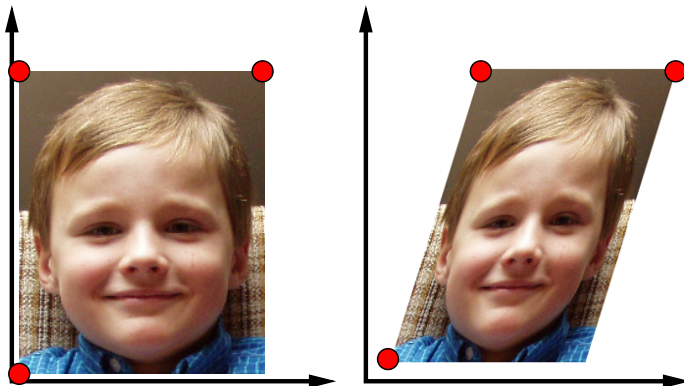
$$x' = a_{11}x + a_{12}y + x_0$$

$$y' = a_{21}x + a_{22}y + y_0$$



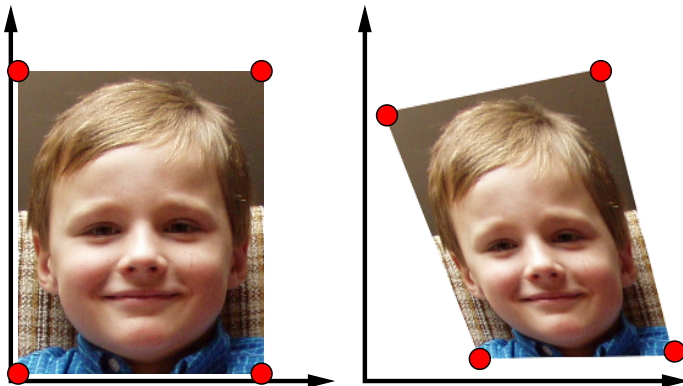
3-point:
(translation + scale + rotation + skew)

$$A = \begin{pmatrix} a_{11} & a_{12} & x_0 \\ a_{21} & a_{22} & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



**4-point:
(homography)**

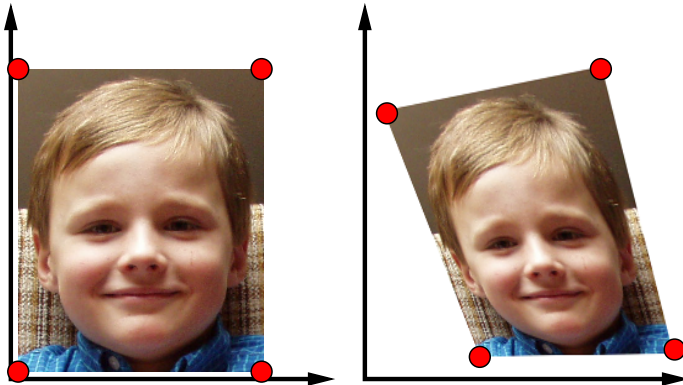
$$x' = \frac{a_{11}x + a_{12}y + x_0}{f_1x + f_2y + 1}$$
$$y' = \frac{a_{21}x + a_{22}y + y_0}{f_1x + f_2y + 1}$$



4-point:
(homography)

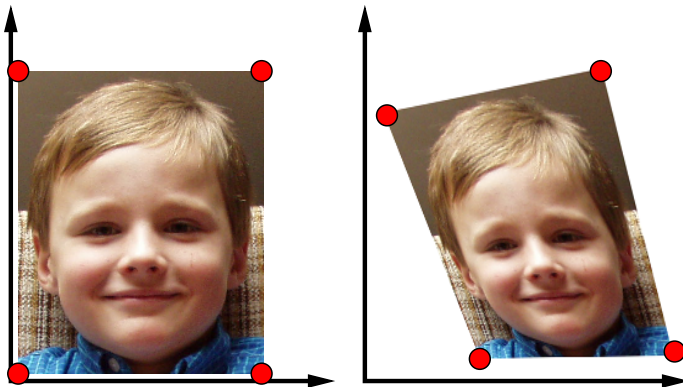
$$(f_1x + f_2y + 1) \cdot x' = a_{11}x + a_{12}y + x_0$$

$$(f_1x + f_2y + 1) \cdot y' = a_{21}x + a_{22}y + y_0$$

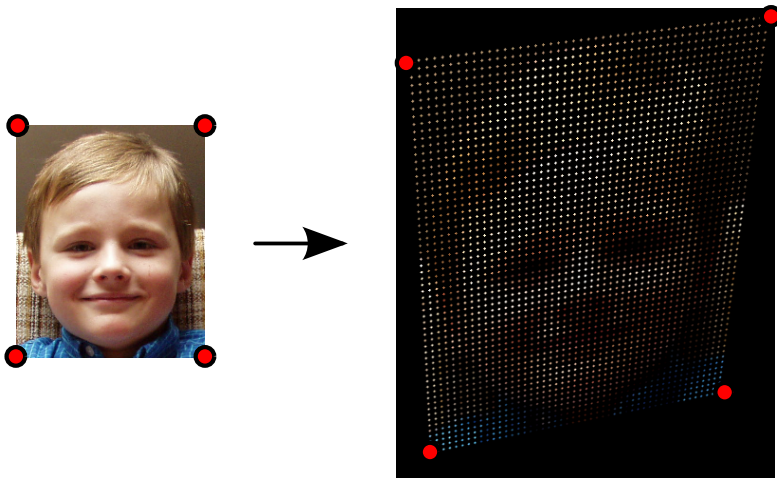


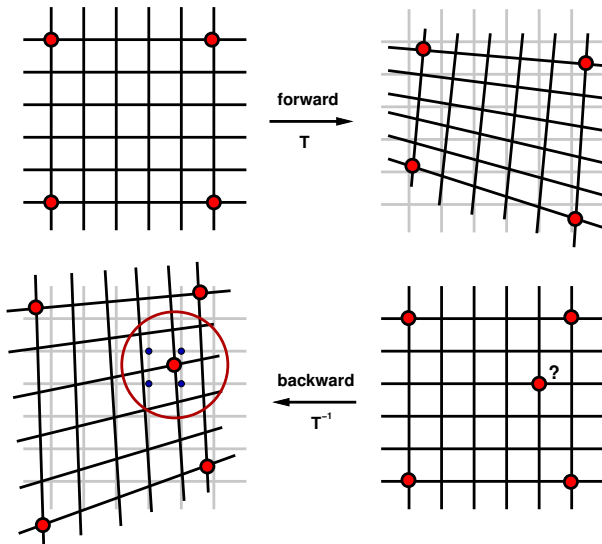
4-point:
(homography)

$$H = \begin{pmatrix} a_{11} & a_{12} & x_0 \\ a_{21} & a_{22} & y_0 \\ f_1 & f_2 & 1 \end{pmatrix}$$



Problem: directly transformed pixels will not fill the target grid.





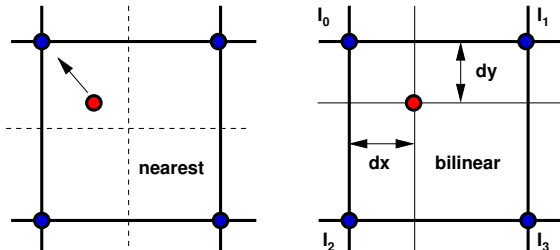
Nearest neighbor:

$$\mathbf{I}[x', y'] = \mathbf{I}[\lfloor x \rfloor, \lfloor y \rfloor]$$

Bilinear interpolation:

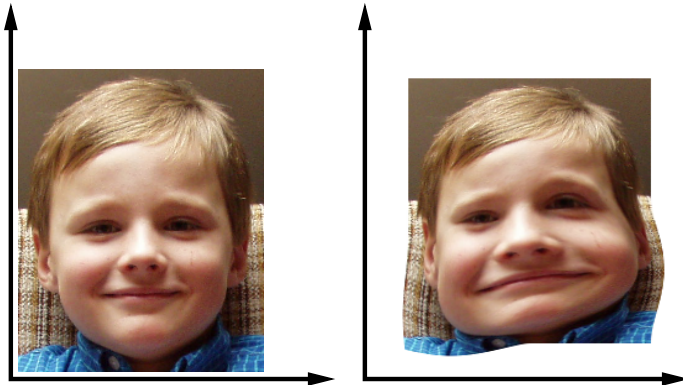
$$\begin{aligned} \mathbf{I}_0 &= \mathbf{I}[\lfloor x \rfloor, \lfloor y \rfloor] & \mathbf{I}_3 &= \mathbf{I}[\lfloor x \rfloor + 1, \lfloor y \rfloor + 1] \\ \mathbf{I}_1 &= \mathbf{I}[\lfloor x \rfloor + 1, \lfloor y \rfloor] & \mathbf{d}_x &= x - \lfloor x \rfloor \\ \mathbf{I}_2 &= \mathbf{I}[\lfloor x \rfloor, \lfloor y \rfloor + 1] & \mathbf{d}_y &= y - \lfloor y \rfloor \end{aligned}$$

$$\mathbf{I}[x', y'] = (\mathbf{I}_0(1 - \mathbf{d}_x) + \mathbf{I}_1\mathbf{d}_x)(1 - \mathbf{d}_y) + (\mathbf{I}_2(1 - \mathbf{d}_x) + \mathbf{I}_3\mathbf{d}_x)\mathbf{d}_y$$

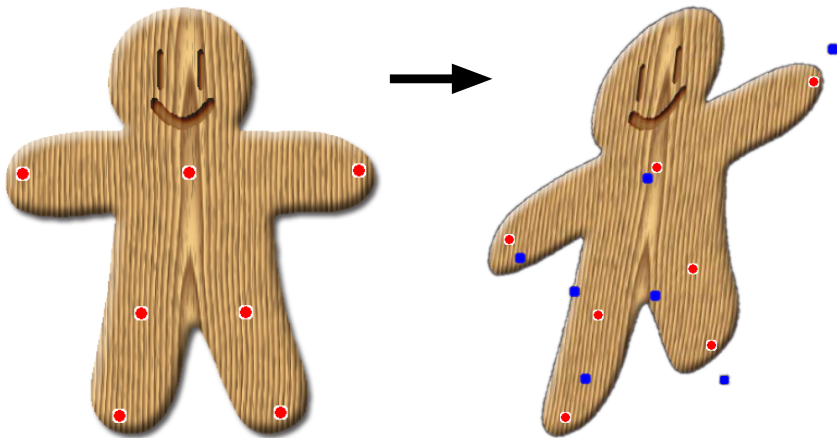


N-point:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$



Least squares (affine):



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$$(\mathbf{A}^*, \mathbf{t}^*) = \arg \min_{\mathbf{A}, \mathbf{t}} \sum_i |\mathbf{p}_i \mathbf{A} + \mathbf{t} - \mathbf{q}_i|^2$$

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$$\mathbf{p}_c = \frac{1}{N} \sum_i \mathbf{p}_i \quad \mathbf{q}_c = \frac{1}{N} \sum_i \mathbf{q}_i$$

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$$\mathbf{p}_c = \frac{1}{N} \sum_i \mathbf{p}_i \quad \mathbf{q}_c = \frac{1}{N} \sum_i \mathbf{q}_i$$

$$\hat{\mathbf{p}}_i = \mathbf{p}_i - \mathbf{p}_c \quad \hat{\mathbf{q}}_i = \mathbf{q}_i - \mathbf{q}_c$$

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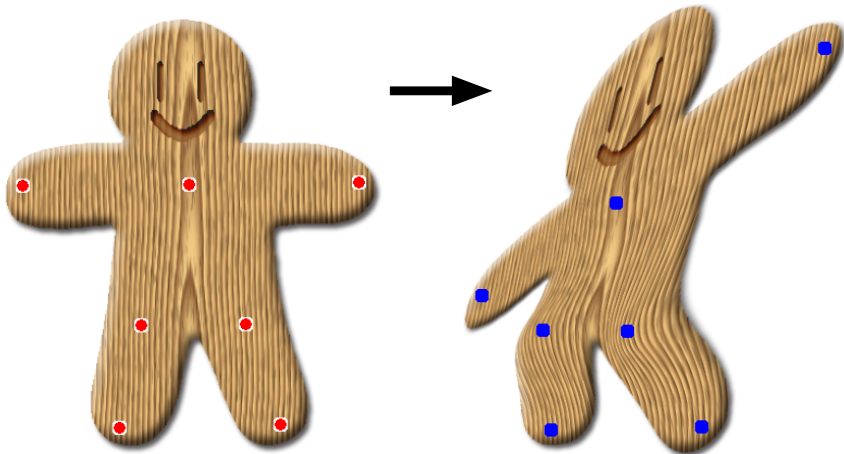
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Moving least squares (affine):



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$$(\mathbf{A}^*, \mathbf{t}^*) = \arg \min_{\mathbf{A}, \mathbf{t}} \sum_i w_i |\mathbf{p}_i \mathbf{A} + \mathbf{t} - \mathbf{q}_i|^2$$

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$$w_i = \frac{1}{(\mathbf{p}_i - \mathbf{v})^{2\alpha}}$$

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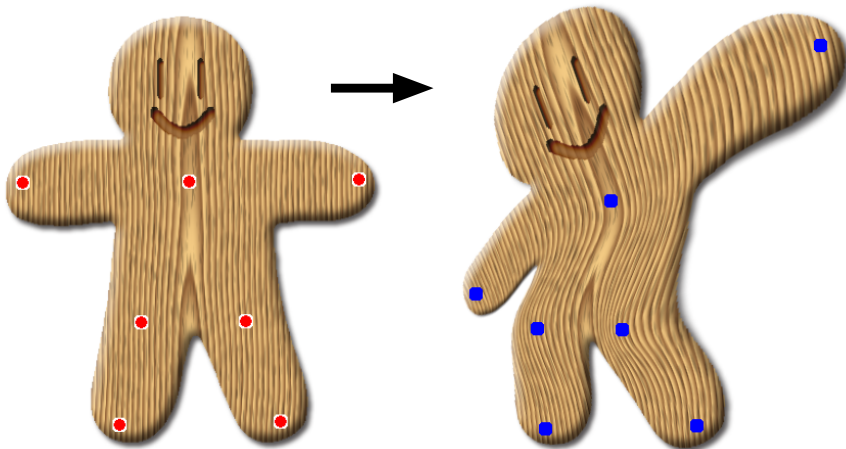
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$$\mathbf{A}^* = \left(\sum_i \hat{\mathbf{p}}_i^T w_i \hat{\mathbf{p}}_i \right)^{-1} \sum_j \hat{\mathbf{p}}_j^T \hat{\mathbf{q}}_j \quad \mathbf{t}^* = \mathbf{q}_c - \mathbf{p}_c \mathbf{A}^*$$

$$\mathbf{p}_c = \sum_i w_i \mathbf{p}_i / \sum_i w_i \quad \mathbf{q}_c = \sum_i w_i \mathbf{q}_i / \sum_i w_i$$

Moving least squares (as-similar-as-possible):



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$$(\mathbf{S}^*, \mathbf{t}^*) = \arg \min_{\mathbf{S}, \mathbf{t}} \sum_i w_i |\mathbf{p}_i \mathbf{S} + \mathbf{t} - \mathbf{q}_i|^2$$

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$$\mathbf{S}^T \mathbf{S} = \lambda^2 \mathbf{I}$$

$$\mathbf{S}^* = \frac{1}{\mu} \sum_i w_i \begin{pmatrix} \hat{\mathbf{p}}_i \\ \hat{\mathbf{p}}_i^\perp \end{pmatrix} (\hat{\mathbf{q}}_i^T \quad \hat{\mathbf{q}}_i^{\perp T})$$

Moving least squares (as-similar-as-possible):

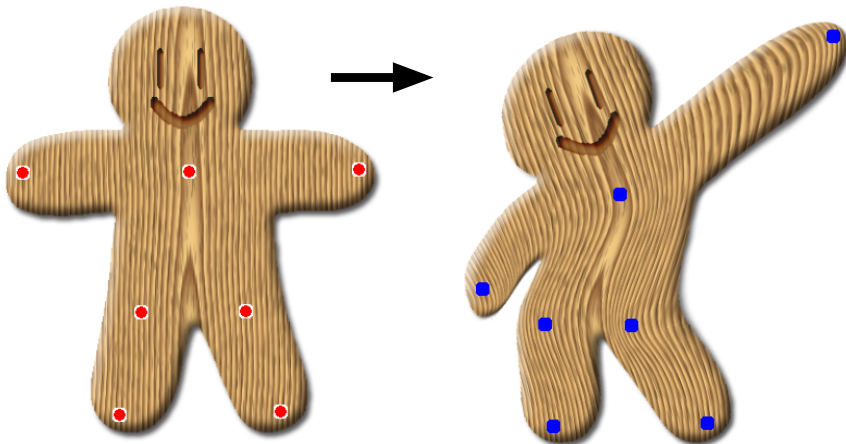
$$(\mathbf{S}^*, \mathbf{t}^*) = \arg \min_{\mathbf{S}, \mathbf{t}} \sum_i \mathbf{w}_i |\mathbf{p}_i \mathbf{S} + \mathbf{t} - \mathbf{q}_i|^2$$

$$\mathbf{S}^T \mathbf{S} = \lambda^2 \mathbf{I}$$

$$\mathbf{S}^* = \frac{1}{\mu} \sum_i w_i \begin{pmatrix} \hat{\mathbf{p}}_i \\ \hat{\mathbf{p}}_i^\perp \end{pmatrix} (\hat{\mathbf{q}}_i^T \quad \hat{\mathbf{q}}_i^{\perp T})$$

$$\mu = \sum_i w_i \hat{\mathbf{p}}_i \hat{\mathbf{p}}_i^T \quad (x, y)^\perp = (y, -x)$$

Moving least squares (as-rigid-as-possible):



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$$(\mathbf{R}^*, \mathbf{t}^*) = \arg \min_{\mathbf{R}, \mathbf{t}} \sum_i w_i |\mathbf{p}_i \mathbf{R} + \mathbf{t} - \mathbf{q}_i|^2$$

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$$\mathbf{R}^T \mathbf{R} = \mathbf{I}$$

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$$\mu = \sqrt{\left(\sum_i w_i \hat{\mathbf{q}}_i \hat{\mathbf{p}}_i^T \right)^2 + \left(\sum_i w_i \hat{\mathbf{q}}_i \hat{\mathbf{p}}_i^{\perp T} \right)^2}$$