Digital Image

(B4M33DZO, Summer 2024)

Lecture 2:

Fourier Transform

https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start

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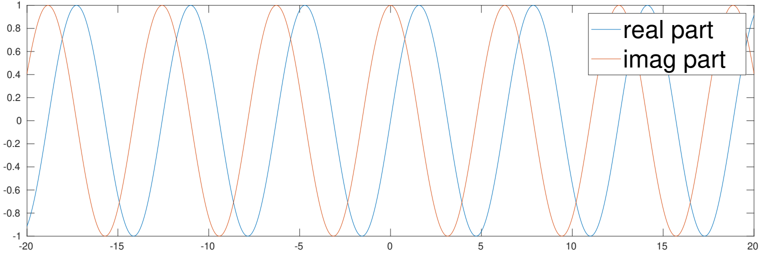
Fourier Transform

$$\textit{time}\{t\} \quad \Longleftrightarrow \quad \textit{frequency}\{u\}$$

$$\text{forward:} \quad F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i u t}\, \mathrm{d}t$$

$$\text{inverse:} \quad f(t) = \int_{-\infty}^{\infty} F(u)\, e^{+2\pi i u t}\, \mathrm{d}u$$

$$\text{basis functions:} \quad e^{2\pi i u t} = \cos(2\pi u t) + i\sin(2\pi u t)$$



Fourier Transform

$$\textit{time}\{t\} \iff \textit{frequency}\{u\}$$

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forward: $$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i u t}\, \mathrm{d}t$$

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basis functions: $$e^{2\pi i u t} = \cos(2\pi u t) + i \sin(2\pi u t)$$

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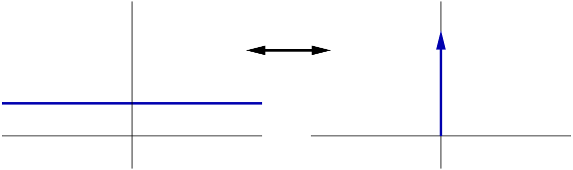
amplitude: $$|F(u)| = \sqrt{\mathrm{Re}(F(u))^2 + \mathrm{Im}(F(u))^2}$$

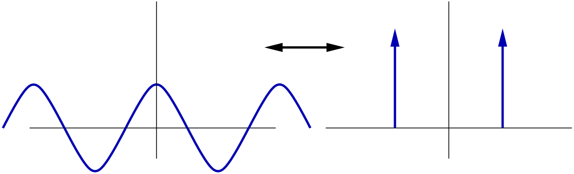
phase: $$\Phi(F(u)) = \tan^{-1}\left(\frac{\mathrm{Im}(F(u))}{\mathrm{Re}(F(u))}\right)$$

Fourier Transform

$$1 \quad \Longleftrightarrow \quad \delta(u)$$

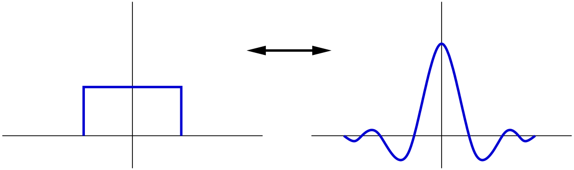


$$\cos(2\pi kx) \quad \Longleftrightarrow \quad \frac{1}{2}\Big(\delta(u+k)+\delta(u-k)\Big)$$

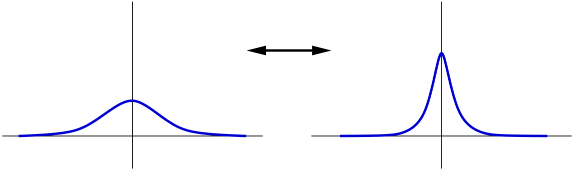


Fourier Transform

$$1(x+k) - 1(x-k) \quad \Longleftrightarrow \quad \frac{1}{\pi u}\sin(2\pi k u)$$



$$\exp(-kx^2) \quad \Longleftrightarrow \quad \sqrt{\frac{\pi}{k}}\exp\left(-\frac{\pi^2}{k}u^2\right)$$



Fourier Transform — Properties

- **Linearity:**

$$a \cdot f(x) + b \cdot f(x) \quad \Longleftrightarrow \quad a \cdot F(u) + b \cdot F(u)$$

- **Parseval's theorem:**

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \,\mathrm{d}u$$

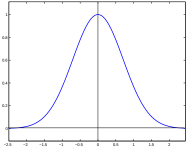
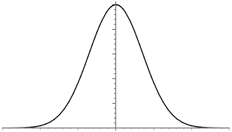
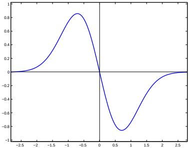
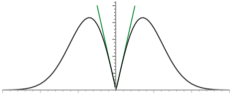
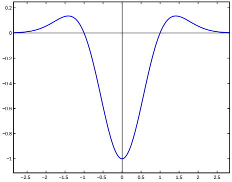
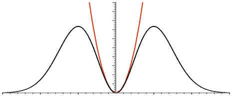
- **Convolution theorem:**

$$\int_{-\infty}^{\infty} f(x) \cdot g(t - x) \,\mathrm{d}x \quad \Longleftrightarrow \quad F(u) \cdot G(u)$$

- **Shift theorem:**

$$\int_{-\infty}^{\infty} f(x - a)\, e^{-2\pi iux} \,\mathrm{d}x = F(u)\, e^{-2\pi iua}$$

Fourier Transform — Derivatives

time	**derivatives**	**frequency**
	**0th:** $f(x) \Longleftrightarrow F(u)$	
	**1st:** $f'(x) \Longleftrightarrow 2\pi iu \cdot F(u)$	
	**2nd:** $f''(x) \Leftrightarrow (2\pi iu)^2 \cdot F(u)$	

Fourier Transform (in 2D)

$2D$ *position* $\{x, y\}$ $\iff$ $2D$ *frequency coords* $\{u, v\}$
(or in polar coordinates: *frequency* (scale of waves) and *orientation* (direction of waves))

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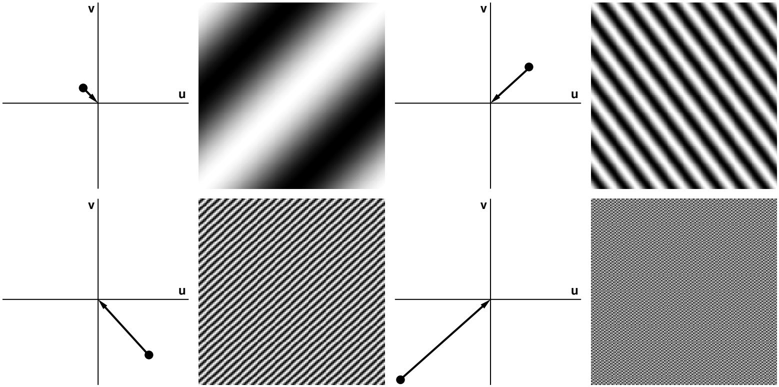
forward: $F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-2\pi i(ux+vy)}\, \mathrm{d}x\, \mathrm{d}y$

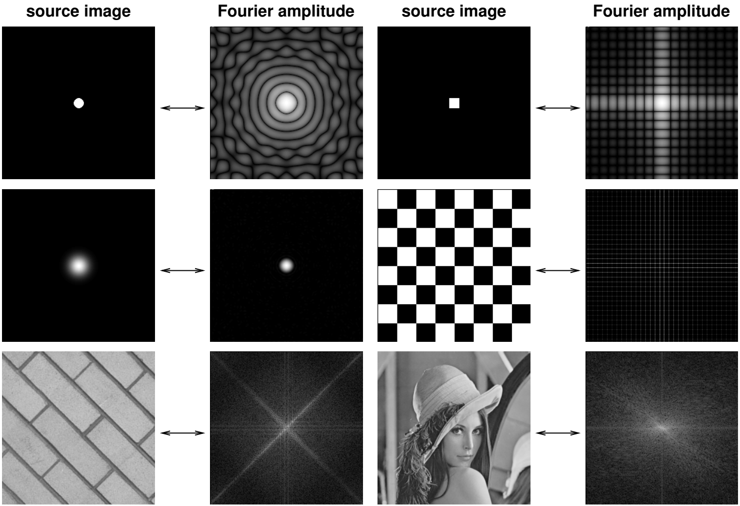
inverse: $f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)\, e^{+2\pi i(ux+vy)}\, \mathrm{d}u\, \mathrm{d}v$

Fourier Transform (in 2D) — Basis Functions

Basis functions (only real part shown):



Fourier Transform (in 2D)



Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT):

$$\text{forward:} \quad F[u,v] = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f[x,y]\, e^{-2\pi i(ux+vy)/N}$$

$$\text{inverse:} \quad f[x,y] = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F[u,v]\, e^{+2\pi i(ux+vy)/N}$$

Computation complexity: $\mathcal{O}(N^4)$. Can we do it faster?

Fast Fourier Transform (FFT)

Computing DFT using Fast Fourier Transform (FFT):

separability: $$F[u,v] = \frac{1}{N}\sum_{x=0}^{N-1} e^{-2\pi i u x/N}\left(\sum_{y=0}^{N-1} f[x,y]\, e^{-2\pi i v y/N}\right)$$

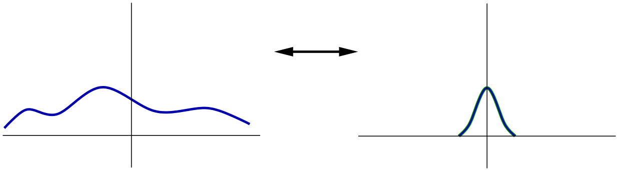
recursion:
$$F[u] = F_{even}[u] + F_{odd}[u]\, e^{-2\pi i u/N},$$
$$F[u+N/2] = F_{even}[u] - F_{odd}[u]\, e^{-2\pi i u/N}.$$

Computation complexity: $\mathcal{O}(N^2 \log N)$.

Sampling Theorem

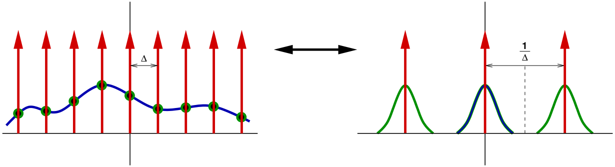
Setting:

Band-limited signal, $f_{max} \leq \frac{1}{2\Delta}$ (Nyquist frequency)



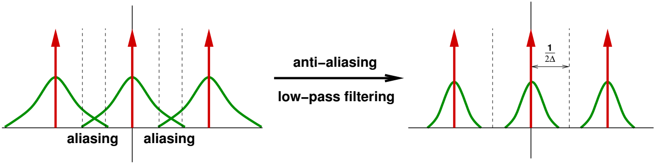
Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$.



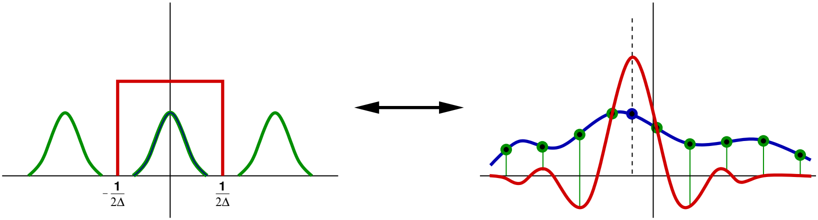
$$s(x) = \sum_k \delta\left(x - k\Delta\right) \iff S(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

$$d(x) = f(x) \cdot s(x) \iff D(u) = (F * S)(u)$$



Sampling Theorem — Signal Reconstruction

Signal reconstruction:



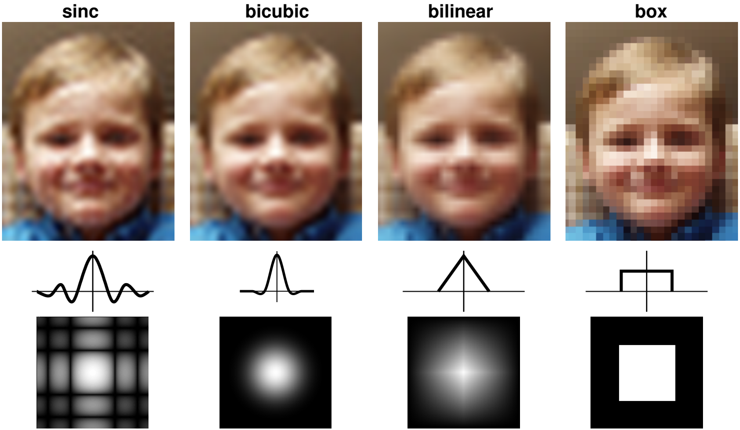
$$\color{red}{B}(u) = 1\left(u + \frac{1}{2\Delta}\right) - 1\left(u - \frac{1}{2\Delta}\right) \quad \Longleftrightarrow \quad \color{red}{b}(x) = \frac{1}{\pi x}\sin\left(\frac{\pi x}{\Delta}\right)$$

$$\color{blue}{F}(u) = \color{green}{D}(u) \cdot \color{red}{B}(u) \quad \Longleftrightarrow \quad \color{blue}{f}(x) = (\color{green}{d} * \color{red}{b})(x)$$

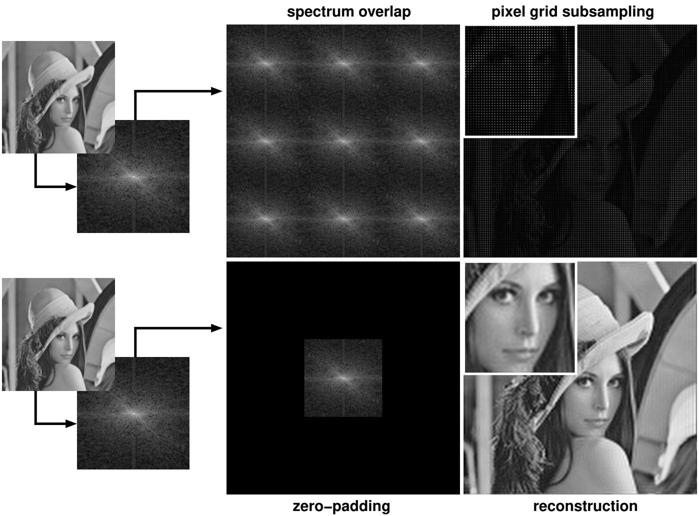
Image Resampling

Problem: convolution with sinc ⇒ time consuming, ringing artifacts.

Sinc approximations with narrow support (bicubic, bilinear, box):



Example

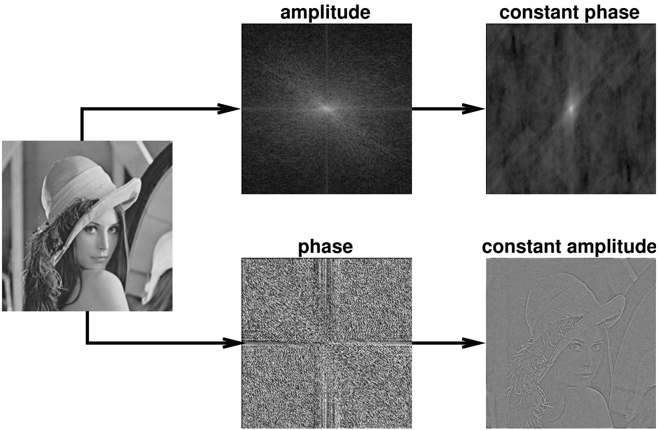

spectrum overlap
pixel grid subsampling
zero–padding
reconstruction

Aliasing Example

Fourier Transform — Amplitude/Phase

Edges with orientation $\arctan(v/u)$ and frequency $\sqrt{u^2 + v^2}$:

Amplitude $\Rightarrow$ intensity Phase $\Rightarrow$ "location"



Keeping only phase (and synthesizing amplitude)

Let us keep the original phase only, and replace the amplitude by A $= f^{-\beta}$, $f = \sqrt{u^2 + v^2}$:

$\beta = 0$

$\beta = 0.4$

$\beta = 0.8$

$\beta = 1$

$\beta = 1.5$

$\beta = 2$