Analysis of the 1st price auction with $n=2$ bidders and uniformly distributed valuations on $[0,1]$

We will prove that (s^*, s^*) is an equilibrium strategy profile, where $s^*(v) = \frac{1}{2} v$, $\forall v \in [0,1]$.

Assume that bidder 1 submits a bid $b_1 \geq 0$, while bidder 2 follows strategy s^*. Then the expected payoff of bidder 1 is

$$\mathbb{E}\left[u_1(b_1, s^*(V_2))\right] =$$

$$= \mathbb{E}\left[u_1\left(b_1, \tfrac{1}{2} V_2\right)\right]$$

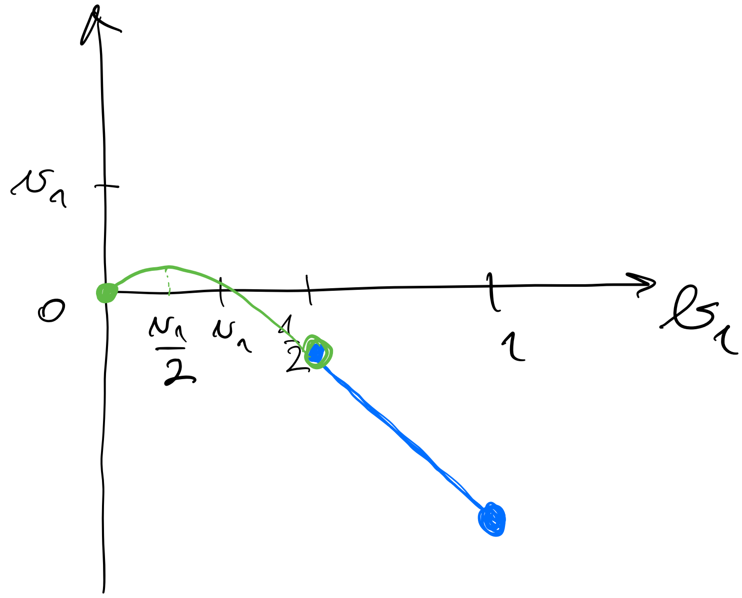
$$= P\left[\tfrac{1}{2} V_2 < b_1\right](v_1 - b_1)$$

$$= P\left[V_2 < 2b_1\right](v_1 - b_1)$$

$$= \min\{2b_1, 1\} \cdot (v_1 - b_1)$$

$$= \begin{cases} \boxed{2b_1(v_1 - b_1)}, & \text{if } 0 \leq b_1 \leq 1/2, \\ \boxed{v_1 - b_1}, & \text{if } 1/2 < b_1 \leq 1. \end{cases}$$

We can plot this function :



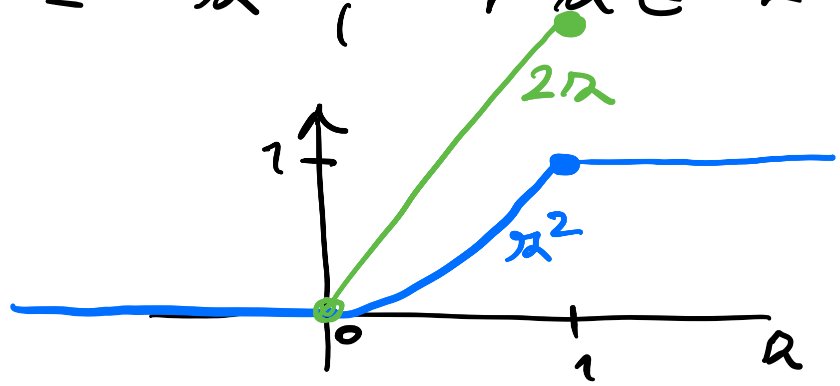
The maximum is at $b_1 = \frac{1}{2} v_1$, so this is the BR strategy equal to s^*.

We will now compute the expected revenue for the seller.
This is equal to

$$\mathbb{E}\left[\max\left(\frac{V_1}{2}, \frac{V_2}{2}\right)\right] = \frac{1}{2}\mathbb{E}\left[\underbrace{\max(V_1, V_2)}_{Z :=}\right]$$

Let's compute the DF of random variable Z :

$$F_Z(x) = P[\max(V_1, V_2) \le x]$$
$$= P[V_1 \le x,\ V_2 \le x]$$
$$= P[V_1 \le x] \cdot P[V_2 \le x]$$
$$= x^2, \quad \forall x \in [0,1],$$



This means that the PDF of Z is $f_Z(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases}$

So the expected revenue is then

$$\frac{1}{2}\mathbb{E}(Z) = \frac{1}{2}\int_0^1 x \cdot (2x)\, dx =$$
$$= \int_0^1 x^2\, dx = 1/3$$

By the revenue equivalence, this coincides with the expected revenue from the 2nd price auction:

$$\mathbb{E}[\min(V_1, V_2)].$$

We will use the identity

$$V_1 + V_2 = \max(V_1, V_2) + \min(V_1, V_2).$$

Then $\mathbb{E}[\min(V_1,V_2)] =$

$$\underbrace{\mathbb{E}[V_1]}_{1/2} + \underbrace{\mathbb{E}[V_2]}_{1/2} - \underbrace{\mathbb{E}[\max(V_1,V_2)]}_{2/3}$$

$= 1/3$

$$\boxed{\text{More about equilibrium strategies in the 1st price auction}}$$

Let $s^*(v) = \mathbb{E}[Y \mid Y < v]$

$$= \frac{1}{G(v)} \int_0^v y\, g(y)\, dy.$$

We can use the integration by parts (per partes) to prove that

$$s^*(v) = v - \int_0^v \frac{G(y)}{G(v)}\, dy.$$

Since $\dfrac{G(y)}{G(v)} < 1 \quad \forall y \in [0, v)$,

the optimal bid satisfies

$$\boxed{s^*(v) < v.}$$

Moreover, since

$$\frac{G(y)}{G(v)} = \left(\frac{F(y)}{F(v)}\right)^{n-1},$$

the degree of shading the bid depends on the number of bidders.

As $n \to \infty$, $\left(\dfrac{F(y)}{F(v)}\right)^{n-1} \to 0$,

and $\boxed{s^*(v) \approx v}$.