The purpose of these notes is to describe the concept of a flow game. Loosely speaking, a flow game is a coalitional game derived from a flow problem in a network. Edges of the network are controlled by players and the worth of each coalition is determined by the maximal flow of the corresponding flow problem. The details follow; see also [1].

1 Flow problem

A *flow network* is a tuple (V, E, v^0, v^1, c), where

(i) (V, E) is a directed graph,

(ii) $v^0 \in V$ is a *source* (no edges in),

(iii) $v^1 \in V$ is a *sink* (no edges out),

(iv) $c\colon E \to \mathbb{R}_0^+$ is a *capacity function*.

Let $N \coloneqq \{1, \ldots, n\}$ be a set of players. Each player controls one or more edges in a flow network. This is captured by a function $I\colon E \to N$, where $I(u, v) = j$ says that edge $(u, v) \in E$ is controlled by player $j \in N$. A *flow problem* is a tuple

$$\mathcal{F} \coloneqq (V, E, v^0, v^1, c, N, I). \tag{1}$$

We can think of a flow network as a communication system between v^0 and v^1, where each link $(u, v) \in E$ connecting two vertices $u \in V$ and $v \in V$ has some capacity $c(u, v) \geq 0$. The main task is to transport as much of certain quantity (information, oil etc.) as possible from source v^0 to sink v^1, while respecting capacities of all links. The additional data of a flow problem $\mathcal{F}$ are the labels of players $I(u, v)$ on edges $(u, v) \in E$. We will see their role in Section 2.

A *flow* for a flow problem $\mathcal{F}$ is a function $f\colon E \to \mathbb{R}_0^+$ such that:

(i) $f(u, v) \leq c(u, v)$ for all $(u, v) \in E$.

(ii) For every vertex $v \in V \setminus \{v^0, v^1\}$,

$$\sum_{\substack{u \in V \setminus \{v\} \\ (u,v) \in E}} f(u, v) = \sum_{\substack{u \in V \setminus \{v\} \\ (v,u) \in E}} f(v, u).$$

A flow transports a certain amount of quantity from v^0 to v^1 such that the capacity of each link is not exceeded and the incoming flow is the same as the outgoing flow. The *magnitude* of a flow f is the total flow $M(f)$ arriving at v^1,

$$M(f) \coloneqq \sum_{\substack{u \in V \setminus \{v^1\} \\ (u,v^1) \in E}} f(u, v^1).$$

It is also true that

$$M(f) = \sum_{\substack{u \in V \setminus \{v^0\} \\ (v^0,u) \in E}} f(v^0, u).$$

The goal is to maximize the magnitude over the set of all flows. It is well-known that the flow whose magnitude is maximal exists and is called the *maximal flow*.

2 Flow game

We will describe a coalitional game in which the worth of each coalition is the magnitude of the maximal flow that can be effectively transported only by the members of the coalition. Let $\mathcal{F}$ be a flow problem (1). For each coalition $A \subseteq N$, let E_A be the set of links controlled by the members of A,

$$E_A \coloneqq \{(u, v) \in E \mid I(u, v) \in A\}.$$

Thus, we obtain a flow problem

$$\mathcal{F}_A \coloneqq (V, E_A, v^0, v^1, c, A, I).$$

Note that the domains of mappings c and I above are restricted to E_A. Since $\mathcal{F}_A$ is a flow problem, there exists a maximal flow for $\mathcal{F}_A$ (see Section 1). A *flow game* is a coalitional game $v_{\mathcal{F}} \colon \mathcal{P}(N) \to \mathbb{R}$ such that, for each $A \in \mathcal{P}(N)$, the worth of A is

$$v_{\mathcal{F}}(A) \coloneqq \text{the magnitude of the maximal flow for } \mathcal{F}_A.$$

The number $v_{\mathcal{F}}(A) \geq 0$ is the maximal flow that can pass from v^0 to v^1 using only the communication links under the control of coalition A.

References

[1] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.