Chicken with one-sided ignorance

Bob

Alice

	x	$1-x$
p	3,3	0,0
$1-p$	4,0	-1,-1

saint

	y	$1-y$
p	3,3	0,4
$1-p$	4,0	-1,-1

the usual Bob

	z	$1-z$
p	3,3	0,9
$1-p$	4,0	-1,-1

madman

In blue are the corresponding mixed strategies.

For Alice, the interim utility = ex ante, specifically:

$$U_1(p,(x,y,z)) = -\frac{2}{3}p(x+y+z) + p + \frac{5}{3}(x+y+z) - 1$$

For Bob, each interim utility is the same as ex post utility:

$$U_2(p,(x,y,z)\,|\,s) = 2px + p + x - 1$$

$$U_2(p,(x,y,z)\,|\,u) = -2py + 5p + y - 1$$

$$U_2(p,(x,y,z)\,|\,m) = -7pz + 10p + z - 1$$

Ex ante:

$$U_2(p,(x,y,z)) = \frac{1}{3}\left[(2x-2y-7z+16)p + x + y + z - 3\right]$$

We will investigate pure BNE.

1) Let $p^* = 1$, that is, Alice slows down.
Then from $U_2(p^*, (x, y, z)) = \frac{1}{3}\left[3x - y - 6z - 3\right]$
we see that Bob's BR is the
behavioral strategy

$$x^* = 1,\ y^* = 0,\ z^* = 0.$$

From $U_1(p, (x^*, y^*, z^*)) = \frac{1}{3}p + \frac{2}{3}$
we see that $p = p^* = 1$ is the BR
of Alice. This implies that
the profile of pure strategies

$$(1, (1, 0, 0))$$

is a pure BNE.

2) Let $p^* = 0$. Then $U_2(p^*, (x, y, z)) =$
$= \frac{1}{3}\left[x + y + z - 3\right]$, so the BR of Bob
is now

$$x^* = y^* = z^* = 1.$$

Since $U_1(p, (1, 1, 1)) = -p + 4$,
so $p^* = 0$ is the BR, and
$(0, (1, 1, 1))$ is also a pure BNE.

Conclusions.

1) Note that in the first pure BNE, the payoff to Alice is 1, which is higher than in the original version of the Chicken game, if she sticks to the same strategy "slow".

2) In the second pure BNE, the payoff to Alice is 4, which is the same as in the original Chicken game when using pure strategy "accelerate".

BoS with mutual ignorance

For the definition see the slides.

Let B denote the type "Bach lover" and σ denote the type "Stravinski lover".

The common prior is

$$\rho(B,B) = \frac{1}{10}, \ \rho(B,\sigma) = \frac{2}{10}, \ \rho(\sigma,\sigma) = \frac{7}{10}.$$

Let $((B,S),(B,S))$ be the pure strategy profile such that

1) Alice plays B if she is B, otherwise S
2) Bob plays B if he is B, otherwise S

We want to check that this is a pure BNE. To this end, we need to compute interim expected utility functions.

Let p and q be the probabilities that Alice plays B if she is β and σ, respectively. Analogously, we use the notation x and y for Bob.

Alice – interim expected utilities

$$U_1(\underline{p},q,x,y\,|\,\beta) = \frac{1}{3}\Big(2px + p(1-x) + (1-p)(1-x)\Big) + \frac{2}{3}\Big(py + 2p(1-y) + (1-p)y\Big)$$

$$= \frac{1}{3}\Big((2x-4y+4)p - x + 2y + 1\Big)$$

$$U_1(p,\underline{q},x,y\,|\,\sigma) = \underbrace{U_1(p,\underline{q},x,\underline{y}\,|\,\sigma,\sigma)}_{\text{ex post}}$$

$$= 2qy - 2q - y + 2$$

Bob – interim expected utilities

$$U_2(p,q,\underline{x},y\,|\,\beta) = \underbrace{U_2(\underline{p},q,\underline{x},y\,|\,\beta,\beta)}_{\text{ex post}}$$

$$= 2px - p + 1$$

$$U_2(p,q,x,\underline{s}\,|\,\sigma) = \frac{1}{9}\left[(-4p+14q-14)y + 2p - 7q + 16\right]$$

We will check that $((B,S),(B,S))$ is indeed a pure BNE.

First, we verify that (B,S) is the best response of Alice:

$$U_1(p,q,\underset{B}{1},\underset{S}{0}\,|\,B) = 2p \Rightarrow \underbrace{p=1}_{B} \text{ is BR}$$

$$U_1(p,q,1,0\,|\,\sigma) = 2-2q \Rightarrow \underbrace{q=0}_{S} \text{ is BR}$$

This means that (B,S) is the BR of Alice. We can check that (B,S) is the BR of Bob, too, in a similar way.

Sheriff's dilemma

A sheriff is facing an armed suspect and each of them must simultaneously decide whether to shoot or not. It is known that the population contains the proportion $\rho \in [0,1]$ of armed criminals.

This can be formulated as a Bayesian game:

innocent

	Sheriff: shoot	Sheriff: not
Suspect: shoot	-3, -1	-1, -2
Suspect: not	-2, -1	0, 0

criminal

	Sheriff: shoot	Sheriff: not
Suspect: shoot	0, 0	2, -2
Suspect: not	-2, -1	-1, 1

Note that pure strategies marked (highlighted) are strongly dominant for the suspect.

We can conjecture that pure strategy s^*:
innocent $\mapsto$ not and criminal $\mapsto$ shoot
is a part of BNE. How should the sheriff respond to that?

We need to compute the ex ante utility of sheriff, where p = Probability (shoot):

$$u_2(s^*, p) = (1-\rho)\cdot(-1)\cdot p + \rho\cdot(-2)\cdot(1-p)$$

$$= \underbrace{(3\rho-1)}_{\text{positive / negative / 0}} p - 2\rho$$

If $\rho > 1/3$, then the BR of sheriff is $p^* = 1$, that is, to shoot.

If $s < 1/3$, then the BR of sheriff is $p^* = 0$, not shoot.
If $s = 1/3$, then any mixed strategy of sheriff is BR.

Conclusion. This implies that the above defined pure strategy s^* of the suspect and the corresponding strategies of sheriff are BNE for any choice of $s \in [0,1]$.

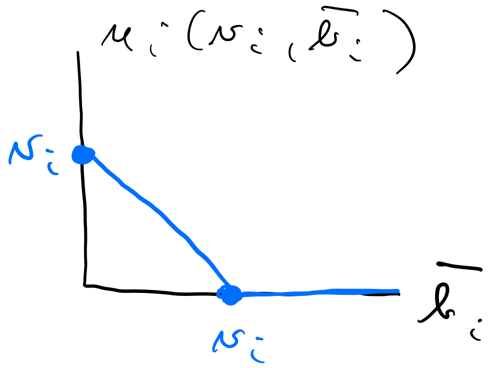
Second-price sealed-bid auction

We already know that the pure strategy v_i (bid your valuation) weakly dominates any other strategy b_i of player i. This means that $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ $\forall b_i \forall b_{-i}$ and $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$ $\forall b_i \exists b_{-i}$.

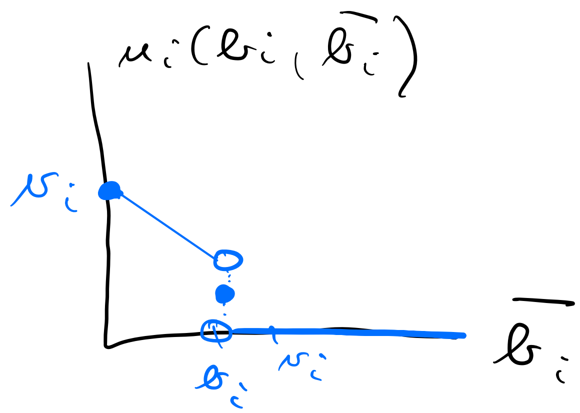
Since the values of u_i depends only on $\overline{b_i} := \max_{j \neq i} b_j$, we can write
$u_i(b_i, \overline{b_i}) := u_i(b_i, b_{-i})$.

The weak domination of v_i is best seen on the graph of functions $u_i(b_i, \overline{b_i})$ in 3 cases.

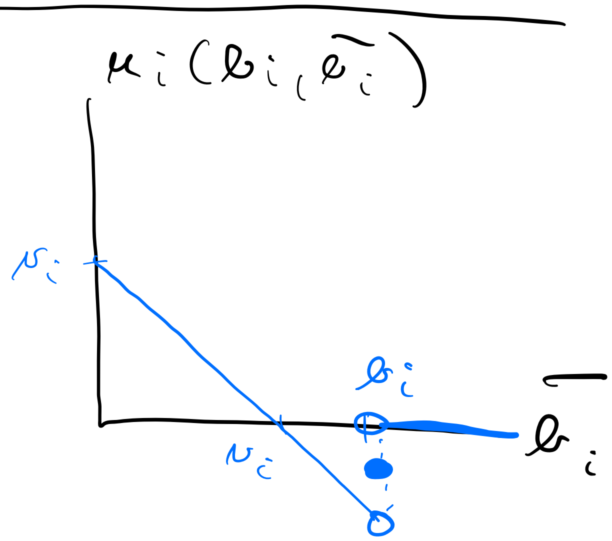
Case 1 ($b_i = v_i$)



Case 2 ($b_i < v_i$)



Case 3 ($b_i > v_i$)



Bayes-Nash equilibrium, equivalently

See the corresponding slide for formulation. Intuitively, it says that an action is best after the information about type is received if and only if it is part of the best conditional plan (behavioral strategy) ahead of time should that information be received.

Proof of 1. $\Rightarrow$ 2.

Let p^* be a BNE. Consider any pure strategy $s_i : T_i \to S_i$ of player i. Then

$$U_i(s_i, p^*_{-i}) = \sum_{t_i} \rho(t_i) \cdot \boxed{U_i(s_i(t_i), p^*_{-i})}$$

$$\leq \sum_{t_i} \rho(t_i) \cdot \boxed{U_i(p^*_i, p^*_{-i})}$$

$$= U_i(p^*_i, p^*_{-i}) = U_i(p^*)$$

where the inequality follows from the inequality of terms in the boxes.

Proof of 2. ⟹ 1.

Conversely, assume that p^* is not BNE. This means that there exists player i, type t_i and an action $a_i \in S_i$ satisfying

$$U_i(p^* | t_i) < U_i(a_i, p^*_{-i} | t_i).$$

We will define a behavioral strategy $\hat{p}_i : T_i \to \Delta_i$ which is a profitable deviation for player i in "ex ante" equilibrium, thus showing that 2. fails. It is not difficult to check that $\hat{p}_i$ can be defined by

$$\hat{p}_i(t_i') = \begin{cases} a_i & t_i' = t_i \\ p^*_i(\cdot | t_i') & t_i' \neq t_i \end{cases}$$

for any type $t_i' \in T_i$.