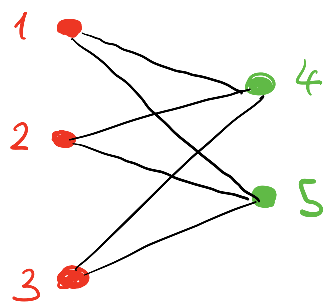
# Example of a zero-sum polymatrix game

$N = \{\underbrace{1,2,3}_{\text{evaders}},\underbrace{4,5}_{\text{inspectors}}\}$




- evaders try to escape through exit points
- every inspector may block only the exit point in his authority while any evader may pick any exit

Let $S_i$ $(i=4,5)$ be the disjoint sets of exit points in the authority of inspector $i$.

Then every evader $i \in \{1,2,3\}$ has a strategy set

$$S_i = S_4 \cup S_5.$$

Further, define payoffs for any evader $i \in \{1,2,3\}$ and any $j \in \{4,5\}$:

Evader: $u_{ij}(s_i, s_j) = \begin{cases} 0 & s_i \notin S_j \\ 1-\delta(s_i,s_j) & s_i \in S_j \end{cases}$

Inspector: $u_{ji}(s_j, s_i) = \delta(s_j, s_i)$

Then $u_i(s_1,\dots,s_5) = 1-\delta(s_i, s_{p(i)})$, where $p(i)$ is the unique index 4 or 5 s.t. $s_i \in S_{p(i)}$,

and

$$u_j(s_1,\dots,s_5) = \left|\{i=1,2,3 \mid s_i = s_j\}\right|$$

$$= \#\text{ evaders caught by } j.$$

The resulting game is polymatrix and constant-sum, since

$$\sum_{i=1}^{5} u_i(s_1,\dots,s_5) = \#\text{ evaders caught} + \#\text{ evaders escaped} = 3$$

## LP for a zero-sum polymatrix game

Let $(p^*, w^*)$ be an optimal solution.

Then $\max_{s_i} U_i(s_i, p^*_{-i}) \leq w_i \quad \forall i \in N,$

and

$$\sum_{i\in N} w_i \geq \sum_{i\in N} \max_{s_i} U_i(s_i, p^*_{-i})$$

$$= \max_{p\in\Delta} \sum_{i\in N} U_i(p_i, p^*_{-i})$$

$$\geq \sum_{i\in N} U_i(p^*) = 0.$$

Therefore, in order for a pair $(p^*, w^*)$ to be optimal, necessarily

$$w_i^* = \max_{s_i} U_i(s_i, p^*_{-i}) = U_i(p^*)$$

for every $i \in N$. The last equality means that $p^*$ is a NE.

## Existence of pure NE for potential games

Let $P : S \to \mathbb{R}$ be the potential. We will show that any

$$s^* \in \arg\max_{s \in S} P(s)$$

is a NE. Indeed, we get

$$P(s^*) \geq P(s_i, s^*_{-i}) \quad \forall s_i$$

$$\Updownarrow$$

$$\underbrace{P(s^*) - P(s_i, s^*_{-i})}_{u_i(s^*) - u_i(s_i, s^*_{-i})} \geq 0$$

$$\Updownarrow$$

$s^*$ is a NE

# Examples of potential games

1) Bach or Stravinski

| | |
|---|---|
| 2,1 | 0,0 |
| 0,0 | 1,2 |

| | |
|---|---|
| 2 | 1 |
| 0 | 2 |

potential

2) Prisoners' dilemma

| | g | s |
|---|---|---|
| g | -1,-1 | -4,0 |
| s | 0,-4 | -3,-3 |

| | |
|---|---|
| 0 | 1 |
| 1 | 2 |

potential

the number of players choosing strategy "s"

3) Congestion games

# FP - failure of convergence

| | d | e | f |
|---|---|---|---|
| a | 0,0 | 1,0 | 0,1 |
| b | 0,1 | 0,0 | 1,0 |
| c | 1,0 | 0,1 | 0,0 |

The unique NE is $(p_1^*, p_2^*)$, where each $p_i^*$ is a uniform probability distribution.

However, the empirical distributions fail to converge when FP starts at $(a, e)$.

# BR dynamics - failure of convergence

| | c | d |
|---|---|---|
| a | 1 | -1 |
| b | -1 | 1 |

Starting from $(a,c)$, the dynamics cycle

$$\begin{tikzcd} (a,c) \arrow[r] & (a,d) \arrow[d] \\ (b,c) \arrow[u] & (b,d) \arrow[l] \end{tikzcd}$$

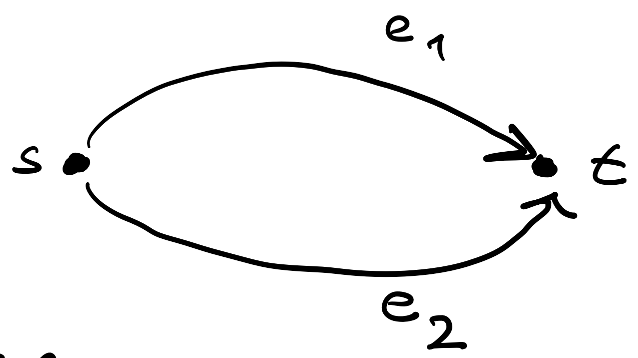
The empirical frequencies correspond asymptotically to the unique NE of the Matching Pennies.

Note that this is no longer true in the game

| | |
|---|---|
| 1000 | -1 |
| -1 | 1 |

in which BR dynamics exhibit the same cycling behavior, but the NE is not uniform !

# Congestion games




Each driver chooses $e_1$ or $e_2$.

Edge latencies when $k$ drivers use it:

$\ell_{e_1}(1) = 1$, $\ell_{e_1}(2) = 3$

$\ell_{e_2}(1) = \ell_{e_2}(2) = 2$

A driver's total cost is the latency of the single edge she picks:

| edges | loads | 1st driver's cost | 2nd driver's cost |
|---|---|---|---|
| $(e_1, e_1)$ | $(2,0)$ | 3 | 3 |
| $(e_2, e_2)$ | $(0,2)$ | 2 | 2 |
| $(e_1, e_2)$ | $(1,1)$ | 1 | 2 |
| $(e_2, e_1)$ | $(1,1)$ | 2 | 1 |

There are 2 NE, $(e_1, e_2)$ and $(e_2, e_1)$.

This is a potential game:

$$P(\sigma) = \sum_{e \in \{e_1, e_2\}} \sum_{k=1}^{n_e(\sigma)} l_e(k)$$

where $\sigma \in \{e_1, e_2\}^2$ and $n_e(\sigma)$ is the number of drivers using edge $e$ under $\sigma$.

For example

$$P(e_1, e_2) = 3.$$