NORMAL-FORM GAMES

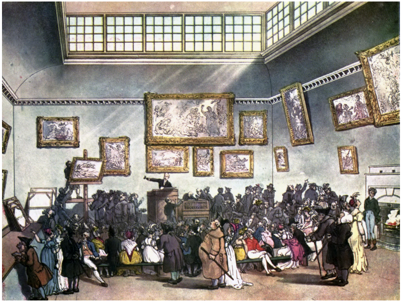
Tomáš Kroupa

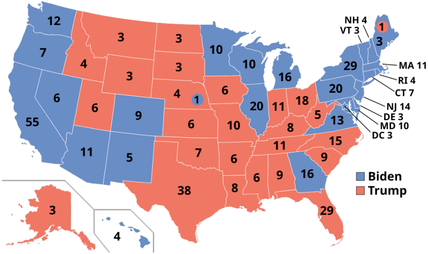
AI Center
Department of Computer Science
Faculty of Electrical Engineering
Czech Technical University in Prague

GAME THEORY

- Mathematical theory of interactive decision-making

Players ⟶ Strategies ⟶ Outcomes ⟶ Utilities





Wikipedia

STORIES OF SUCCESS

Theoretical foundations

- *On the theory of games of strategy* (von Neumann, 1928)
- Nash equilibrium (1951)
- Complexity class PPAD (Papadimitriou, 1984)

Mechanism and security design

- Auction design (Vickrey, 1961)
- *Using Game Theory for Los Angeles Airport Security* (2009)

Solving large games

- *DeepStack: Expert-level AI in heads-up no-limit poker* (2017)
- AlphaStar (DeepMind, 2019)

CLASSIFICATION OF GAMES

Game forms

- normal
- extensive
- coalitional

Information about the game

- complete
- incomplete

Dynamics

- static
- dynamic

Number of strategies

- finite
- infinite

Utility functions

- zero-sum
- general-sum

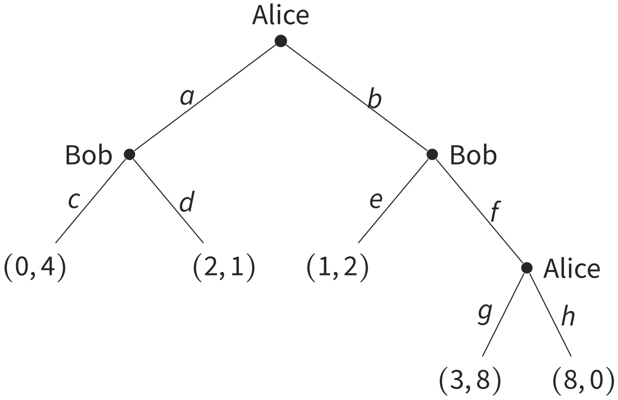
Information during the play

- perfect
- imperfect

Competitive/cooperative

A GAME WITH PERFECT INFORMATION

Alice meets Bob for breakfast/lunch. Bob chooses for breakfast Starbucks or McDonald's, and a lunch venue. They might go to a Czech restaurant or a pizzeria, where Alice chooses between Sicilian or Neapolitan pizza.



Which venue will be chosen?

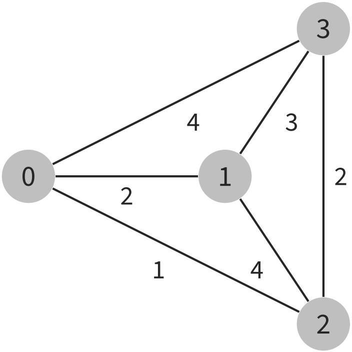
FIRST-PRICE SEALED-BID AUCTION

- A single item is up for auction
- Each buyer submits a bid simultaneously in a sealed envelope
- The item is given to the highest bidder who pays her bid

What is the optimal bidding strategy for each buyer?

COOPERATIVE GAME

- Cities 1,2, and 3 need to connect to the provider of energy 0
- The graph shows costs of pairwise connections



How to allocate the costs?

OUTLINE OF THE COURSE

Topic	*Lectures*	*Why?*
normal-form games	5	fundamentals
extensive-form games	2	card and board games
Bayesian games	3	auctions
cooperative games	3	cost allocation, voting

NORMAL-FORM GAMES

NORMAL-FORM GAMES

Definition

Normal-form (or strategic) game is defined by:

- *Player set* $N = \{1, \dots, n\}$, where $n \in \mathbb{N}$
- *Strategy set* S_i for each player $i \in N$
- *Utility function*

$$u_i \colon \mathbf{S} \to \mathbb{R}$$

  for each player $i \in N$, where $\mathbf{S} = S_1 \times \cdots \times S_n$

- Players select strategies $s_1 \in S_1, \dots, s_n \in S_n$ simultaneously
- The resulting utility of player i is $u_i(s_1, \dots, s_n)$

EXAMPLE

Prisoner's dilemma

The police make the following offer to two prisoners. If one squeals on the other that both committed the serious crime, then the confessor will be set free and the other will spend 4 years in jail. If both confess, then they will each get the 3-year sentence. If both stay quiet, then they will each spend 1 year in jail for the minor offense.

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

TWO-PLAYER ZERO-SUM GAMES

- $N = \{1, 2\}$
- $u_1 + u_2 = 0$

The value of $u_1(s_1, s_2)$ is being tugged away at from two sides, by player 1 who wants to maximize it, and by player 2 who wants to minimize it.

von Neumann (1928)

Matching pennies

Each player places a penny on a table, heads up or tails up. If the pennies match, player 1 wins; otherwise player 2 wins.

	h	t
h	1	−1
t	−1	1

FIRST-PRICE SEALED-BID AUCTION

One object is for sale. Each buyer i submits a bid $b_i \geq 0$ simultaneously in a sealed envelope. The item is given to the *highest* bidder who pays her bid.

- Every bidder attaches a private value $v_i > 0$ to the item.
- The winner is selected uniformly at random from the k highest bidders.

Strategic game representation

Strategy sets $S_i = [0, \infty)$ and utility functions

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & b_i < \overline{b}_i, \\ (v_i - b_i)/k & b_i = \overline{b}_i, \\ v_i - b_i & b_i > \overline{b}_i, \end{cases}$$

where $\overline{b}_i := \max_{j \neq i} b_j$.

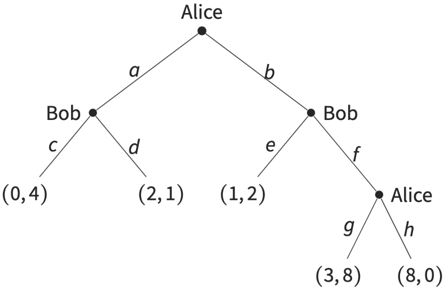
SECOND-PRICE SEALED-BID AUCTION

The rules are the same as in the first-price auction, except that the winner pays the price equal to the *second-highest* bid.

Strategic game representation

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & b_i < \overline{b}_i, \\ (v_i - \overline{b}_i)/k & b_i = \overline{b}_i, \\ v_i - \overline{b}_i & b_i > \overline{b}_i. \end{cases}$$

FROM EXTENSIVE-FORM GAMES TO STRATEGIC GAMES



Bob

Alice		*ce*	*cf*	*de*	*df*
	ag	0, 4	0, 4	2, 1	2, 1
	ah	0, 4	0, 4	2, 1	2, 1
	bg	1, 2	3, 8	1, 2	3, 8
	bh	1, 2	8, 0	1, 2	8, 0

SOLUTION CONCEPTS FOR STRATEGIC GAMES

RATIONALITY

A person's behavior is rational if it is in his best interests, given his information. R. Aumann (2006)

- The utilitaristic concept of rationality imposes no restrictions on the norms of human behavior

Both parties deprecated war; but one would make war rather than let the nation survive; and the other would accept war rather than let it perish. And the war came. A. Lincoln (1865)

- We assume that rational players maximize utility

DOMINATION OF STRATEGIES

Which strategies a player never uses?

Definition

Let $s, t \in S_i$ be strategies of player i. We say that

- s strictly dominates t if $u_i(s, \mathbf{s}_{-i}) > u_i(t, \mathbf{s}_{-i})$ for every $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$,
- s weakly dominates t if $u_i(s, \mathbf{s}_{-i}) \geq u_i(t, \mathbf{s}_{-i})$ for any $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$ and $u_i(s, \mathbf{s}_{-i}) > u_i(t, \mathbf{s}_{-i})$ for some $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$.

- A rational player never uses a strictly dominated strategy

ITERATED ELIMINATION OF STRICTLY DOMINATED STRATEGIES

- $\ell := 0, S_i^0 := S_i$ for each $i \in N$
- Repeat
  1. For each $i \in N$, pick any *strictly* dominated strategy $d_i^\ell \in S_i^\ell$ in the *subgame* given by $S_1^\ell, \ldots, S_n^\ell$
  2. $S_i^{\ell+1} := S_i^\ell \setminus \{d_i^\ell\}$ for each $i \in N$
  3. $\ell := \ell + 1$
- Until no strictly dominated strategy is found for any player

The strategies that survived all rounds of the algorithm are rationalizable.

EXAMPLE

Bob

		c	d	e
Alice	a	1, 0	1, 2	0, 1
	b	0, 3	0, 1	2, 0

1. Bob is rational.

	c	d
a	1, 0	1, 2
b	0, 3	0, 1

2. Alice is rational and she knows 1.

	c	d
a	1, 0	1, 2

3. Bob is rational and he knows 2.

	d
a	1, 2

EXAMPLES

Matching pennies

	h	t
h	1	-1
t	-1	1

Every strategy is rationalizable.

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

Every strategy is rationalizable.

Prisoner's dilemma

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

Only s is rationalizable.

Coordination game

	a	b
a	$2, 2$	$0, 0$
b	$0, 0$	$1, 1$

Every strategy is rationalizable.

RATIONALIZABILITY

Properties of iterated elimination

- The resulting sets S_i^ℓ do not depend on the *order* of elimination
- However, the order of elimination is crucial when the algorithm is modified to remove *weakly* dominated strategies!

- A unique dominating strategy might exist in special games

TRUTHFUL BIDDING IN SECOND-PRICE AUCTIONS

Strategic game representation

Strategy sets $S_i = [0, \infty)$ and utility functions

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & b_i < \overline{b}_i, \\ (v_i - \overline{b}_i)/k & b_i = \overline{b}_i, \\ v_i - \overline{b}_i & b_i > \overline{b}_i. \end{cases}$$

Proposition

The strategy v_i *weakly* dominates every other strategy $b_i \in S_i$ of player i.

NASH EQUILIBRIUM

A strategy profile in which no player has an incentive to deviate, assuming the strategies of all other players remain unchanged.

Definition

A pure strategy profile $\mathbf{s}^* \in \mathbf{S}$ is a Nash equilibrium if for every player $i \in N$

$$u_i(\mathbf{s}^*) \geq u_i(s, \mathbf{s}^*_{-i}) \qquad \text{for all } s \in S_i.$$

NASH EQUILIBRIUM, EQUIVALENTLY

Which strategy will player i adopt as a reply to the opponents' strategies?

- Best response mapping

$$\mathrm{BR}_i(\mathbf{s}_{-i}) = \operatorname*{argmax}_{s \in S_i} u_i(s, \mathbf{s}_{-i}) \qquad \text{for all } \mathbf{s}_{-i} \in \mathbf{S}_{-i}$$

- Best response to $\mathbf{s}_{-i}$ is any strategy $s \in \mathrm{BR}_i(\mathbf{s}_{-i})$

Proposition

The following are equivalent for a strategy profile $\mathbf{s}^* \in \mathbf{S}^*$.

1. $\mathbf{s}^*$ is a Nash equilibrium.
2. $s_i^* \in \mathrm{BR}_i(\mathbf{s}_{-i}^*)$, for each $i \in N$.

NASH EQUILIBRIUM – EXAMPLES

Matching pennies

	h	t
h	1	-1
t	-1	1

There is no NE in pure strategies.

Bach or Stravinski

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

(B,B), (S,S) are the only pure NE.

Prisoner's dilemma

	q	s
q	$-1,-1$	$-4,0$
s	$0,-4$	$-3,-3$

(s,s) is the only NE.

Coordination game

	a	b
a	$2,2$	$0,0$
b	$0,0$	$1,1$

(a,a),(b,b) are the only pure NE.

PARETO OPTIMALITY

Which strategy profiles are socially efficient?

Definition

A strategy profile $\mathbf{t} \in \mathbf{S}$ is Pareto optimal if there is no $\mathbf{s} \in \mathbf{S}$ such that $u_i(\mathbf{s}) \geq u_i(\mathbf{t})$ for all $i \in N$ and there is some $i \in N$ such that $u_i(\mathbf{s}) > u_i(\mathbf{t})$.

- No player's utility can be increased without simultaneously decreasing the utility to another player
- Individual preferences of players are ignored

HOW TO FIND A PARETO OPTIMAL PROFILE?

Maximizing social welfare

Define $w\colon \mathbf{S} \to \mathbb{R}$ as

$$w(\mathbf{s}) = \sum_{i \in N} u_i(\mathbf{s}), \quad \mathbf{s} \in \mathbf{S}.$$

Every maximizer of w is Pareto optimal.

- Pareto optimal strategy profile exists in every *finite* strategic game
- All strategy profiles in a zero-sum game are Pareto optimal

PARETO OPTIMALITY – EXAMPLES

Matching pennies

	h	t
h	1	-1
t	-1	1

Every profile is Pareto optimal.

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

(B, B), (S, S) are Pareto optimal.

Prisoner's dilemma

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

Only (s, s) is *not* Pareto optimal.

Coordination game

	a	b
a	$2, 2$	$0, 0$
b	$0, 0$	$1, 1$

(a, a) is Pareto optimal.

CONCLUSIONS

1. No definitive solution using pure strategies in the Matching Pennies
2. Every Nash equilibrium involves only rationalizable strategies
3. All strategies might be rationalizable while there is a single NE
4. Socially efficient solution is not necessarily reachable