Extensive-Form Games

Branislav Bošanský

Artificial Intelligence Center,
Department of Computer Science,
Faculty of Electrical Engineering,
Czech Technical University in Prague

branislav.bosansky@agents.fel.cvut.cz

October 19, 2021

Previously ... on computational game theory.

1. Mixed Strategies
2. Minimax Theorem
3. Mathematical programs for computing various solution concepts in normal-form games

Today, you should learn ...

1. How to formulate a game in the extensive form?
2. How to transform games between the normal form and the extensive form?
3. How to solve extensive-form games with perfect information?

Comparison of Different Equilibria

The expected values of these solution concepts coincide in zero-sum games.

For two-players general-sum games, the solution concepts are fundamentally different:

- CE is a probability distribution over possible outcomes, desired outcome is sampled and corresponding actions are sent to players as recommendations; following the recommendations is best response for the players.
- NE is a pair of mixed strategies (probability distributions over pure strategies)—one for each player—such that these mixed strategies are best responses to the strategy of the opponent.
- SE is a pair of strategies where leader's mixed strategy is such a public commitment that maximizes the outcome of the leader while the follower plays the best response.

Comparison of Different Equilibria

For two-players general-sum games, the solution concepts are fundamentally different:

- CE can be computed in polynomial time with a single LP. Even finding some specific CE is polynomial.
- NE can be computed in exponential time with a single LCP. Finding some specific NE is NP-complete.
- SSE can be computed in polynomial time with multiple LPs.

Beyond Normal-Form Representations

One representation does not rule them all

Beyond Normal-Form Representations

Beyond Normal-Form Representations

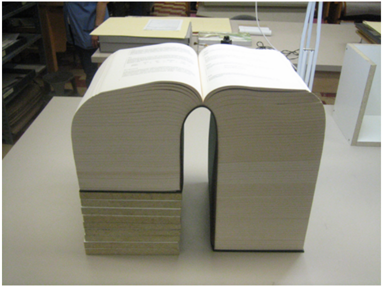
We can represent such dynamic scenarios using the normal-form representation.

A strategy in a dynamic game has to reflect all possible situations we can encounter in a game (including due to the moves by the opponent and/or stochastic events). Therefore, we need to have an action prescribed to be played in each situation that can happen.

The obvious drawback of using this representation is that there is exponentially many possible strategies given a description of the game.

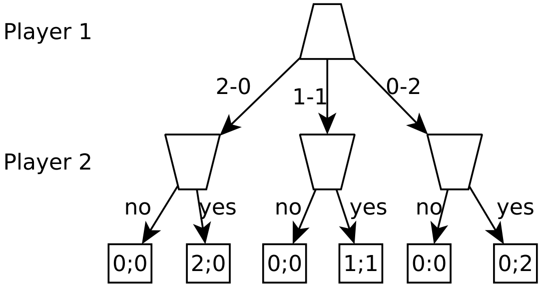
Strategies in Sequential Games

A strategy in a dynamic game has to reflect all possible situations we can encounter in a game (including due to the moves by the opponent and/or stochastic events). Therefore, we need to have an action prescribed to be played in each situation that can happen.



Extensive-Form Representation

We can use a more compact representation that is suitable for finite games termed *extensive-form games*.



Extensive-Form Games (EFGs)

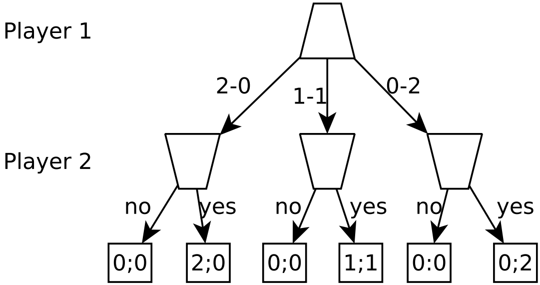
Formal Definition:

- players $\mathcal{N} = \{1, 2, \ldots, n\}$
- actions $\mathcal{A}$
- choice nodes (histories) $\mathcal{H}$
- action function $\chi : \mathcal{H} \to 2^{\mathcal{A}}$
- player function $\rho : \mathcal{H} \to \mathcal{N}$
- terminal nodes $\mathcal{Z}$
- successor function $\varphi : \mathcal{H} \times \mathcal{A} \to \mathcal{H} \cup \mathcal{Z}$
- utility function $u = (u_1, u_2, \ldots, u_n)\,;\; u_i : \mathcal{Z} \to \mathbb{R}$

A pure strategy of player i in an EFG is an assignment of an action for each state where player i acts

$$S_i := \prod_{h \in \mathcal{H}, \rho(h) = i} \chi(h)$$

Strategies in EFGs

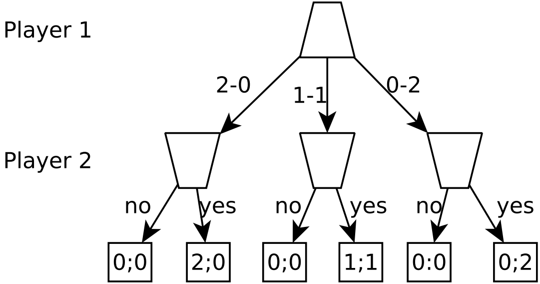


What are actions and strategies in this game?

$$\mathcal{A}_1 = \{2-0, 1-1, 0-2\};\ \mathcal{S}_1 = \{2-0, 1-1, 0-2\}$$

$$\mathcal{A}_2 = \{no, yes\};\ \mathcal{S}_2 = \{(no, no, no), (no, no, yes), \ldots, (yes, yes, yes)\}$$

Strategies in EFGs

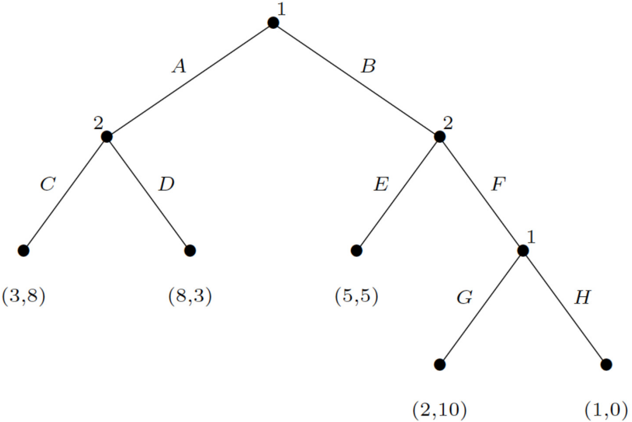


We can replace the function χ by multiplying actions so that an action can be applied only in a single state.

$$\mathcal{A}_2 = \{no_{\{2-0\}}, yes_{\{2-0\}}, no_{\{1-1\}}, yes_{\{1-1\}}, no_{\{0-2\}}, yes_{\{0-2\}}\};$$

$$\mathcal{S}_2 = \{(no_{\{2-0\}}, no_{\{1-1\}}, no_{\{0-2\}}), \ldots, (yes_{\{2-0\}}, yes_{\{1-1\}}, yes_{\{0-2\}})\}$$

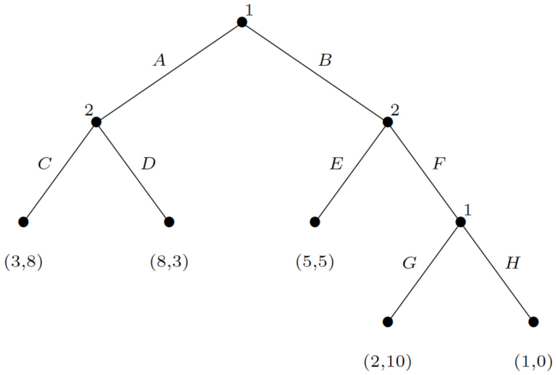
Strategies in EFGs



What are actions and strategies in this game?

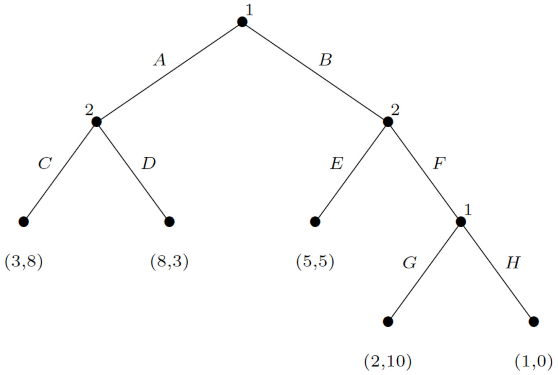
$$\mathcal{S}_1 = \{(A, G), (A, H), (B, G), (B, H)\}$$
$$\mathcal{S}_2 = \{(C, E), (C, F), (D, E), (D, F)\}$$

Induced Normal Form



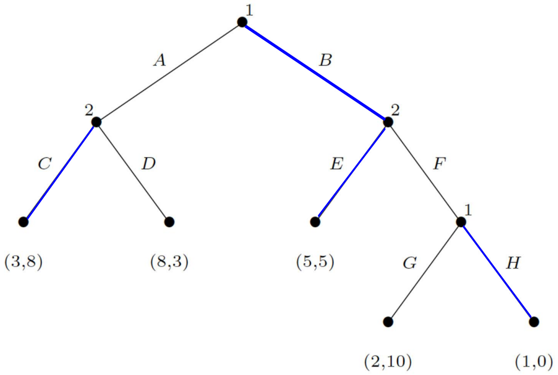
	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	$(3, 8)$	$(3, 8)$	$(8, 3)$	$(8, 3)$
(A, H)	$(3, 8)$	$(3, 8)$	$(8, 3)$	$(8, 3)$
(B, G)	$(5, 5)$	$(2, 10)$	$(5, 5)$	$(2, 10)$
(B, H)	$(5, 5)$	$(1, 0)$	$(5, 5)$	$(1, 0)$

Nash Equilibria in EFGs



	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	$(3,8)$	$(\mathbf{3},\mathbf{8})$	$(8,3)$	$(8,3)$
(A,H)	$(3,8)$	$(\mathbf{3},\mathbf{8})$	$(8,3)$	$(8,3)$
(B,G)	$(5,5)$	$(2,10)$	$(5,5)$	$(2,10)$
(B,H)	$(\mathbf{5},\mathbf{5})$	$(1,0)$	$(5,5)$	$(1,0)$

Nash Equilibria in EFGs - threats



	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	$(3, 8)$	$(\mathbf{3}, \mathbf{8})$	$(8, 3)$	$(8, 3)$
(A, H)	$(3, 8)$	$(\mathbf{3}, \mathbf{8})$	$(8, 3)$	$(8, 3)$
(B, G)	$(5, 5)$	$(2, 10)$	$(5, 5)$	$(2, 10)$
(B, H)	$(\mathbf{5}, \mathbf{5})$	$(1, 0)$	$(5, 5)$	$(1, 0)$

Nash Equilibria in EFGs

Not all Nash strategies are entirely “sequentially rational” in EFGs. Off the equilibrium path, the players may use irrational actions.

We use *refinements of NE* in EFGs to avoid this. The best known (for EFGs with perfect information) is **Subgame-perfect equilibrium**.

Definition (Subgame)

Given a perfect-information extensive-form game G, the subgame of G rooted at node h is the restriction of G to the descendants of h. The set of subgames of G consists of all of subgames of G rooted at some node in G.

Subgame-Perfect Equilibrium (SPE)

Definition (Subgame-perfect equilibrium)

The *subgame-perfect equilibria (SPE)* of a game G are all strategy profiles s such that for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.

function $\textsc{BackwardInduction}$(node h)
    if $h \in \mathcal{Z}$ **then**
        return $u(h)$
    end if
    best_util $\leftarrow \infty$
    for all $a \in \chi(h)$ **do**
        util_at_child $\leftarrow \textsc{BackwardInduction}(\varphi(h, a))$
        if $\textit{util\_at\_child}_{\rho(h)} > \textit{best\_util}_{\rho(h)}$ **then**
            best_util $\leftarrow$ *util_at_child*
        end if
    end for
end function

Subgame-Perfect Equilibrium (SPE)

This is the same algorithm (in principle) that you know as Minimax (or Alpha-Beta pruning, or Negascout) and works (in general) for n-player games.

Corollary

Every extensive-form game with perfect information has at least one Nash equilibria in pure strategies that is also a Subgame-perfect equilibrium.

Is this correct? We have seen examples of games that do not have pure NE.

Not every game can be represented as an EFG with perfect information.

EFGs with Chance

We introduce a new "player" termed chance (or Nature) that plays using a fixed randomized strategy.

Formal Definition:

- players $\mathcal{N} = \{1, 2, \ldots, n\} \cup \{c\}$
- actions $\mathcal{A}$
- choice nodes (histories) $\mathcal{H}$
- action function $\chi : \mathcal{H} \to 2^{\mathcal{A}}$
- player function $\rho : \mathcal{H} \to \mathcal{N}$
- terminal nodes $\mathcal{Z}$
- successor function $\varphi : \mathcal{H} \times \mathcal{A} \to \mathcal{H} \cup \mathcal{Z}$
- stochastic transitions $\gamma : \Delta\{\chi(h) \mid h \in \mathcal{H}, \rho(h) = c\}$
- utility function $u = (u_1, u_2, \ldots, u_n)$; $u_i : \mathcal{Z} \to \mathbb{R}$

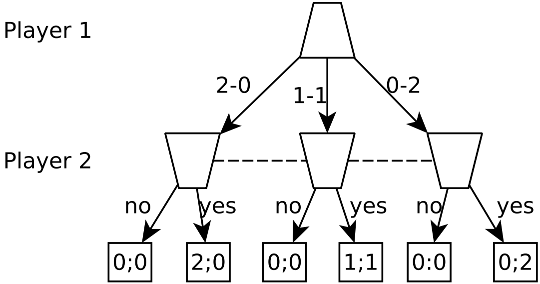
EFGs with Imperfect Information

When players are not able to observe the state of the game perfectly, we talk about *imperfect information games*. The states that are not distinguishable to a player belong to a single *information set*.

Formal Definition:

- $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{H}, \mathcal{Z}, \chi, \rho, \varphi, \gamma, u)$ is a perfect-information EFG.
- $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$ where $\mathcal{I}_i$ is a set of equivalence classes on choice nodes of a player i with the property that $\rho(h) = \rho(h') = i$ and $\chi(h) = \chi(h')$, whenever $h, h' \in I$ for some information set $I \in \mathcal{I}_i$
- we can use $\chi(I)$ instead of $\chi(h)$ for some $h \in I$

Strategies in EFGs with Imperfect Information



What are actions and strategies in this game?

$$\mathcal{A}_1 = \{2-0, 1-1, 0-2\};\ \mathcal{S}_1 = \{2-0, 1-1, 0-2\}$$

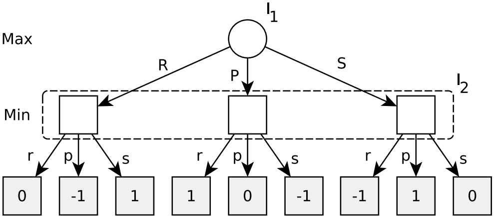
$$\mathcal{A}_2 = \{no, yes\};\ \mathcal{S}_2 = \{no, yes\}$$

Strategies in EFGs with Imperfect Information

There are no guarantees that a pure NE exists in imperfect information games.

Every finite game can be represented as an EFG with imperfect information.

	R	**P**	**S**
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$



Strategies in EFGs with Imperfect Information

Mixed strategies are defined as before as a probability distribution over pure strategies.

There are also other types of strategies in EFGs, namely *behavioral strategies*:

- A *behavioral strategy* of player i is a product of probability distributions over actions in each information set

$$\beta_i : \prod_{I \in \mathcal{I}_I} \Delta(\chi(I))$$

There is a broad class of imperfect-information games in which the expressiveness of mixed and behavioral strategies coincide – *perfect recall games.* Informally, no player forgets any information she previously knew in these games.

Perfect Recall in EFGs

Definition

Player i has perfect recall in an imperfect-information game G if for any two nodes h,h' that are in the same information set for player i, for any path $h_0, a_0, \ldots, h_n, a_n, h$ from the root of the game tree to h and for any path $h_0, a'_0, \ldots, h'_m, a'_m, h'$ from the root to h' it must be the case that:

1. $n = m$
2. for all $0 \leq j \leq n$, h_j and h'_j are in the same equivalence class for player i
3. for all $0 \leq j \leq n$, if $\rho(h_j) = i$, then $a_j = a'_j$

Definition

We say that an EFG has a *perfect recall* if all players have perfect recall. Otherwise we say that the game has an *imperfect recall*.

Regret

The concept of regret is useful when the other players are not completely malicious.

	L	**R**
U	$(100, a)$	$(1-\varepsilon, b)$
D	$(2, c)$	$(1, d)$

Definition (Regret)

A player i's *regret* for playing an action a_i if the other agents adopt action profile a_{-i} is defined as

$$\left[\max_{a'_i \in \mathcal{A}_i} u_i(a'_i, a_{-i})\right] - u_i(a_i, a_{-i})$$

Regret

Definition (MaxRegret)

A player is *maximum regret* for playing an action a_i is defined as

$$\max_{a_{-i} \in \mathcal{A}_{-i}} \left(\left[\max_{a'_i \in \mathcal{A}_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

Definition (MinimaxRegret)

Minimax regret actions for player i are defined as

$$\arg\min_{a_i \in \mathcal{A}_i} \max_{a_{-i} \in \mathcal{A}_{-i}} \left(\left[\max_{a'_i \in \mathcal{A}_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

How to solve an EFG with imperfect information?

Does a backward induction work?

Does a limited-lookahead search work?

Existing algorithms:

- algorithms based on linear programming
- algorithms based on no-regret learning (reinforcement learning)
- algorithms based on convex optimization