ALTERNATIVES TO NASH EQUILIBRIUM

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CORRELATED EQUILIBRIUM

MOTIVATION

- While the Nash equilibrium assumes that players act independently, people often condition their choices on shared signals:
  - an app's recommendation
  - a traffic light
  - a price announcement
  - a weather forecast
- Correlated equilibrium captures realistic coordination that the Nash equilibrium often misses

COORDINATION INCREASES WELFARE

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

- Two pure NE: (B, B) and (S, S)
- The mixed NE: $p_1^*(B) = p_2^*(S) = 2/3$ with utility $2/3$ for either player

How to choose between (B, B) and (S, S)?

1. Mediator generates the outcome randomly, $p(B, B) = p(S, S) = 0.5$
2. Each player receives private recommendation which action to play
3. Following the recommended actions yields a utility of $3/2$ for each

METAGAME FOR CORRELATION OF ACTIONS

Extensive-form game with imperfect information $\Gamma(p)$

1. The mediator uses a probability distribution p over $\mathbf{S} = S_1 \times \cdots \times S_n$ to generate randomly an action profile $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbf{S}$
2. The mediator tells each player i only s_i
3. Each player i is free to choose any action $s_i' \in S_i$
4. The resulting utility is $u_i(s_1', \ldots, s_n')$

STRATEGIES IN THE METAGAME

- A strategy of player i in game $\Gamma(p)$ is a mapping $\sigma_i \colon S_i \to S_i$ from private recommendations to individual actions
- In particular, the strategy *follow the recommendation* is given by

$$\sigma_i^*(s_i) := s_i \qquad \text{for all } s_i \in S_i$$

- The expected utility of player i under $(\sigma_1^*, \ldots, \sigma_n^*)$ is

$$\sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(\mathbf{s}_{-i} \mid s_i) \cdot u_i(s_i, \mathbf{s}_{-i}) \qquad \text{for all } s_i \in S_i$$

Players follow the recommendations as long as they have no incentive to deviate, given their knowledge of the signal.

Definition

A correlated equilibrium (CE) in a strategic game is a probability distribution p over $\mathbf{S}$ such that $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium in $\Gamma(p)$:

$$\sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(\mathbf{s}_{-i} \mid s_i) \cdot u_i(s_i', \mathbf{s}_{-i}) \quad \leq \quad \sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(\mathbf{s}_{-i} \mid s_i) \cdot u_i(s_i, \mathbf{s}_{-i}),$$

for every player i and every $s_i, s_i' \in S_i$ such that $p(s_i) > 0$.

CORRELATED EQUILIBRIUM, EQUIVALENTLY

Proposition

The following are equivalent for a probability distribution p over $\mathbf{S}$.

1. *p is a correlated equilibrium.*
2. *For each player i and all $s_i, s_i' \in S_i$,*

$$\sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(s_i, \mathbf{s}_{-i}) \cdot u_i(s_i', \mathbf{s}_{-i}) \quad \leq \quad \sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(s_i, \mathbf{s}_{-i}) \cdot u_i(s_i, \mathbf{s}_{-i}).$$

EXAMPLE

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

$$p(B,S) \leq 2p(B,B)$$
$$2p(S,B) \leq p(S,S)$$
$$2p(S,B) \leq p(B,B)$$
$$p(B,S) \leq 2p(S,S)$$

Selected correlated equilibria:

1. $p(B,B) = \alpha, \quad p(S,S) = 1 - \alpha,$ for any $\alpha \in (0,1)$
2. $p(B,B) = 1$
3. $p(S,S) = 1$
4. $p(B,B) = p(S,S) = 2/9, \quad p(B,S) = 4/9, \quad p(S,B) = 1/9$

NASH EQUILIBRIUM INDUCES CORRELATED EQUILIBRIUM

Proposition

Any NE $(p_1^*, \dots, p_n^*)$ of a strategic game induces a CE p^* such that

$$p^*(\mathbf{s}) = \prod_{i \in N} p_i^*(s_i) \qquad \forall \mathbf{s} \in \mathbf{S}.$$

- Since correlated equilibrium is more general than Nash equilibrium, the former may be computationally tractable
- Computing a single CE can be formulated as an LP problem with $|S_1 \times \cdots \times S_n|$ variables

COMPUTATION OF CE

Select the CE maximizing the social welfare

$$\sum_{i \in N} \sum_{\mathbf{s} \in \mathbf{S}} p(\mathbf{s}) u_i(\mathbf{s}).$$

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

$$\text{Maximize} \quad 3p(B,B) + 3p(S,S)$$

subject to

$$p(B,S) \leq 2p(B,B)$$
$$2p(S,B) \leq p(S,S)$$
$$2p(S,B) \leq p(B,B)$$
$$p(B,S) \leq 2p(S,S)$$

The optimal solution: $p(B,B) = \alpha, \quad p(S,S) = 1 - \alpha, \quad \text{for any } \alpha \in [0,1]$

STACKELBERG EQUILIBRIUM

MOTIVATION

- One agent (leader) commits to an action, others (followers) react
  - defender $\Rightarrow$ attackers
  - platform $\Rightarrow$ users
  - price-making firm $\Rightarrow$ competitive fringe
- Computationally tractable and deployed in practice
  - security games (U.S. airport and wildlife protection)
  - patrolling
- We focus on the 2-player case (one leader and one follower)

PUBLIC COMMITMENT TO AN ACTION

Example (Conitzer, 2006)

	c	d
a	$2, 1$	$4, 0$
b	$1, 0$	$3, 1$

- Strategy profile (a, c) is the only NE

The row player (leader) publicly commits to:

1. Action b, the column player (follower) plays d and utilities are $(3, 1)$
2. Mixed strategy $p_1(a) = p_1(b) = 1/2$, then the follower's best responses are c and d since $U_2(p_1, c) = U_2(p_1, d) = 1/2$, and each yields different utility for the leader: $U_1(p_1, c) = 3/2, U_1(p_1, d) = 7/2$

TWO-PLAYER STACKELBERG GAME

Player 1 (leader) and player 2 (follower) interact as follows:

1. The leader publicly commits to a mixed strategy $p_1 \in \Delta_1$.
2. The follower then selects a pure strategy $s_2 \in \mathrm{BR}_2(p_1)$.

Bilevel optimization

The leader wants to solve the problem

$$\max_{p_1 \in \Delta_1} U_1(p_1, s_2)$$

depending on

$$s_2 \in \mathrm{BR}_2(p_1) = \operatorname*{argmax}_{s_2' \in S_2} U_2(p_1, s_2')$$

which is typically non-unique. We need a tie-breaking rule to select s_2.

TIE-BREAKING RULES

1. If $|\mathrm{BR}_2(p_1)| = 1$ for every $p_1 \in \Delta_1$, the leader solves

$$\max_{p_1 \in \Delta_1} U_1(p_1, s_2) \qquad \text{where } \mathrm{BR}_2(p_1) = \{s_2\}$$

2. Otherwise we assume that the follower breaks ties
   - to the disadvantage of the leader
   - in favor of the leader

WEAK AND STRONG STACKELBERG EQUILIBRIUM

The follower picks $s_2 \in \mathrm{BR}_2(p_1)$

1. to the disadvantage of the leader:

$$\max_{p_1 \in \Delta_1} \min_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2)$$

2. in favor of the leader:

$$\max_{p_1 \in \Delta_1} \max_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2)$$

Definition

1. Weak SE (p_1^*, s_2^*) is a solution to the 1st problem.
2. Strong SE (p_1^*, s_2^*) is a solution to the 2nd problem.

WEAK SE MAY NOT EXIST

Example

$$\begin{array}{c|c|c|}
 & c & d \\
a & 2,1 & 4,0 \\
b & 1,0 & 3,1
\end{array}$$

$$p_1 \coloneqq p_1(a)$$

$$\mathrm{BR}_2(p_1) = \begin{cases} d & 0 \le p_1 < 1/2 \\ \{c, d\} & p_1 = 1/2 \\ c & 1/2 < p_1 \le 1 \end{cases}$$

1. Weak SE doesn't exist since there is no maximizer of function

$$p_1 \in [0,1] \quad \mapsto \min_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2) = \begin{cases} p_1 + 3 & 0 \le p_1 < 1/2 \\ p_1 + 1 & 1/2 \le p \le 1 \end{cases}$$

2. Strong SE for the leader is given by $p_1^* = 1/2$

HOW TO COMPUTE STRONG SE?

$$\max_{p_1 \in \Delta_1} \max_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2) = \max_{s_2 \in S_2} \max_{\substack{p_1 \in \Delta_1 \\ s_2 \in \mathrm{BR}_2(p_1)}} U_1(p_1, s_2)$$

Algorithm based on LP

- For each $s_2 \in S_2$ solve the LP:

$$\begin{aligned} \max \quad & U_1(p_1, s_2) \\ \text{subject to} \quad & U_2(p_1, s_2) \geq U_2(p_1, t_2) \qquad \forall t_2 \in S_2 \\ & p_1 \in \Delta_1 \end{aligned}$$

- Strong SE p_1^* is the optimal solution for an LP with the maximal value

SE IN TWO-PLAYER ZERO-SUM GAMES

Proposition

In any two-player zero-sum game, weak SE and strong SE coincide, and both are equal to the set of NE.

- In a two-player zero-sum game, whether a player publicly discloses their strategy or not is inconsequential
- This stands in stark contrast with general-sum games