Surname and name: ________________________________

Task	1	2	3	4	5	*Total*
Maximum	10	10	5	5	10	40
Points						

1. Consider a two-player zero-sum game given by the matrix of utilities of the row player:

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

   (a) (6 pts) Compute the maximin strategy of row player and the value of the game.

   (b) (2 pts) What would be the weak Stackelberg equilibrium with the row player as the leader?

   (c) (2 pts) Change a single entry in the matrix so that the game will have a Nash equilibrium in pure strategies.

Solution:

(a) The linear program with variables $(p, v) \in \mathbb{R}^2$ is $\max v$ s.t. $v \in \mathbb{R}, 0 \leq p \leq 1$, $p \geq v$, $-2p+3(1-p) \geq v$. It can be easily solved graphically. The feasible set is a polyhedron with vertices $(\frac{1}{2}, \frac{1}{2}), (0,0), (1,-2)$, which implies that the optimal solution is $(p^*, v^*) = (\frac{1}{2}, \frac{1}{2})$. The maximin strategy is $(p^*, 1-p^*) = (\frac{1}{2}, \frac{1}{2})$ and the value of the game is $v^* = \frac{1}{2}$.

(b) The leader plays the maximin strategy from (a) and the follower uses the best response strategy, which can be both the first and second column.

(c) The matrix game

$$\begin{bmatrix} 1 & -2 \\ \mathbf{1} & 3 \end{bmatrix}$$

has a unique Nash equilibrium in pure strategies in which the row player plays the second row and the column player uses the first column.

2. (a) (2 pts) What is the general way to transform any n-player normal-form game into an extensive-form game? What is the amount of infosets in this transformed extensive-form game?

   (b) (3 pts) Consider a two-player zero-sum extensive-form game with perfect recall and without chance. Suppose that players take turns and in each decision node, they can play one of A actions and the game terminates after each player plays d actions. Throughout the game, no additional information is being revealed to any player (so each player does not know which actions the opponent played). What is the amount of infosets, sequences and pure strategies in such a game?

(c) (5 pts) Consider the following two-player card game. The game begins with a mandatory bet of 1 chip for both players. After the bet, both players obtain 1 card randomly chosen from the deck of 4 cards containing one Js, two Qs, and one K. The players know their own card but do not observe the card of the opponent. After the cards are dealt, player 1 decides if he wants to continue playing by betting another chip or end the game by folding. If he decides to bet, player 2 also decides between betting 1 chip or folding. If both players decide to bet, the cards are revealed and the player with the higher card wins all the chips currently in game. If either of players decides to fold, he immediately loses and all the chips currently in game are won by his opponent. If the players have cards of the same value, the result is a tie and the chips are split evenly between the players. Write down amount of histories (decision nodes), infosets, terminal histories and sequences for each player in such a game. What would be the amount of variables and constraints in Sequence-form Linear program, that has player 2 part of Nash Equilibrium strategy as part of the solution. Explain your answers in detail.

Solution:

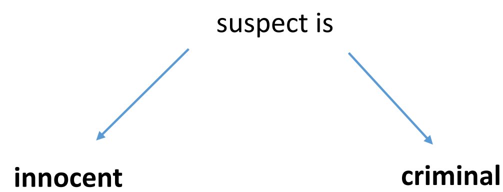
1. Each player will have a single infoset with amount of actions equal to amount of its pure strategies in the normal-form game. Then the terminal utility is assigned based pure strategy combination from the utility matrix. Such game has n infosets.

2. Each player starts with single infoset, because of perfect recall, each additional layer in which player acts has A-times more infosets than previous one. Last decision is after $d-1$ actions. For each player it is $|I| = 1 + A + A^2 \cdots A^{d-1} = \sum_{i=0}^{d-1} A^{d-1}$ infosets. Starting with an empty sequence, each decision node create A new sequences. $|\Sigma| = 1 + A + A^2 + \cdots A^d = \sum_{i=0}^{d} A^i$ for each player. Pure strategy needs to assign an action to each infoset. The result for each player is then $A^|I|$.

3. The game can either start with chance node that gives cards to each player, or it has 2 subsequent chance nodes. In case of single chance node, this chance has 7 possible outcomes (JQ, JK, QJ, QQ, QK, KJ, KQ). Then the player either bets or folds. Each fold create terminal history, each bet create the second player decision. Second players bet or fold both create terminal history. Amount of histories is then $|H| = 1+7+7+7+14 = 36$. In case of 2 chances, the first chance has 3 possibilities. After J or K for first player the second chance has 2 outcomes, after Q it has 3. Then the game proceeds similarly. $|H| = 1+3+(2+3+2)+7+7+14 = 39$. For both approaches, there are $|Z| = 21$ terminal histories. There are $|I| = 3$ infosets for each player based on the card the player received. Since each player plays at most a single action and in each infoset chooses one of two actions, the resulting amount of sequences for each player is $|\Sigma| = 1 + 2|I|$. The Linear program has $\Sigma| + |I|$ variables (for each sequence a single variable and for each opponent's infoset a variable) and $|I| + 1 + 2|I| + |\Sigma|$ constraints (for each player's infoset a single one, plus the empty sequence and for each opponents action in each infoset a single one and each realization plan has to be greater than zero).

3. (5 pts) A sheriff faces an armed suspect. Both players must simultaneously decide whether to

shoot or not. The suspect is a criminal (probability p) or innocent (probability $1 - p$). The sheriff would rather shoot if the suspect shoots, but not if the suspect does not. The criminal suspect would rather shoot even if the sheriff does not, as the criminal would be caught if he does not shoot. The innocent suspect would rather not shoot even if the sheriff shoots.



innocent (rows: *suspect*, columns: *sheriff*)

	Shoot	Not
Shoot	$-3, -1$	$-1, -2$
Not	$-3, -2$	$0, 0$

criminal (rows: *suspect*, columns: *sheriff*)

	Shoot	Not
Shoot	$-0, -0$	$2, -3$
Not	$-2, -1$	$-1, 1$

1. What is the best strategy of criminal suspect and the resulting payoff?
2. What is the best strategy of innocent suspect and the resulting payoff?
3. What is the best strategy of sheriff and the resulting payoff? How does it depend on p?
4. Is there a pure Bayes-Nash equilibrium? If so, which one and how does it depend on p?

Solution:

4. (5 pts) Consider a combinatorial Vickerey-Clarke-Groves (VCG) auction with three bidders $\{a, b, c\}$, two goods $\{g_1, g_2\}$ and the following valuations of bundles:

bundle	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_a	2	1	7
v_b	2	5	7
v_c	3	2	5

(a) List the bids the agents would place in this auction.
(b) Which would be the resulting allocation of the goods?
(c) What ammounts would the agents pay?
(d) Would the resulting allocation of the goods be *efficient*? Justify your answer.
(e) Would the outcome of the auction be *revenue maximizing*? Justify your answer.

Solution:

5. (a) (5 pts) Let v be coalitional game with the player set N. For some player $i \in N$ it is true that $v(A \cup i) = v(A) + v(i)$ for each coalition $A \subseteq N$ not containing i. Show that the Shapley value of player i is $v(i)$.

   (b) (3 pts) Let v be a 3-player simple game with $v(1) = v(2) = v(3) = v(12) = 0$, and $v(13) = v(23) = v(123) = 1$. For each player $i = 1, 2, 3$ decide if i is a null player/dictator/vetoer, and compute the Shapley-Shubik index of i.

   (c) (2 pts) Are there supermodular games with empty cores? Prove or show a counterexample.

Solution:

(a)

$$\varphi_i^S(v) = \sum_{i \notin A \subseteq N} \frac{|A|!(n - |A| - 1)!}{n!} \underbrace{(v(A \cup i) - v(A))}_{v(i)} = v(i) \underbrace{\sum_{i \notin A \subseteq N} \frac{|A|!(n - |A| - 1)!}{n!}}_{1}$$

(b) There is no dictator. Player 3 is a vetoer. Players 1 and 2 are neither vetoers nor null players. Further, $\varphi_1^S(v) = \frac{1}{6}$, which implies (symmetry) that $\varphi_2^S(v) = \frac{1}{6}$, so (efficiency) $\varphi_3^S(v) = \frac{4}{6}$. (c) No. The core of every supermodular game is the convex hull of its marginal vectors.