Surname and name: ______________________________

Task	1	2	3	4	5	*Total*
Maximum	10	10	5	5	10	40
Points						

1. Consider the two-player strategic game with the utility bimatrix

$$\begin{bmatrix} 3,5 & 0,0 \\ 0,0 & 5,3 \end{bmatrix}$$

where the strategy sets are $S_1 = \{a, b\}$ (row player) and $S_2 = \{c, d\}$ (column player).

(a) (1 pts) Explain why this game has at least Nash equilibrium in mixed strategies.

(b) (4 pts) Find a strong Stackelberg equilibrium. The leader is the row player.

(c) (5 pts) Find at least one correlated equilibrium.

Solution:

(a) This is true by the Nash theorem.

(b) Let p be the probability that the leader plays strategy a. We obtain two linear programs. The first is $\max 3p$ s.t. $5p \geq 3(1-p)$, $0 \leq p \leq 1$ and the second is $\max 5(1-p)$ s.t. $3(1-p) \geq 5p$, $0 \leq p \leq 1$. Since the optimal value 5 is higher for the second LP and its optimal solution is $p^* = 0$, the strong SE is pure, (b, d).

(c) The set of all correlated equilibria is the convex polytope of probability distributions $p\colon S_1 \times S_2 \to [0,1]$ in $\mathbb{R}^4$ determined by the linear inequalities $5p(a,d) \leq 3p(a,c)$, $3p(b,c) \leq 5p(b,d)$, $3p(b,c) \leq 5p(a,c)$, $5p(a,d) \leq 3p(b,d)$. For example, it suffices to put $p(a,d) = p(b,c) = 0$, $p(a,c) = t \in [0,1]$, and $p(b,d) = 1-t$.

2. (a) (2 pts) What is the behavioral strategy and what is the realization plan? Write down a formula to extract behavioral strategies from realization plans and conversely.

(b) (3 pts) Consider a two-player zero-sum game with **perfect recall** without any chance element, where players take turns and each player has exactly A actions in each decision node. Furthermore, game terminates after both players play exactly d actions. What is the least and most amount of infosets in such a game? Explain your answer in detail.

(c) (5 pts) Consider the following two-player card game. The game begins with a mandatory bet of 1 chip for both players. After the bet, both players obtain 1 card randomly chosen from the deck of 6 cards containing two Js, one Q, two Ks and one A. The players know their own card but do not observe the card of the opponent. After the cards are dealt, player 1 decides if he wants to continue playing by betting another chip or end the game by folding. If he decides to bet, player 2 also decides between betting 1 chip or folding. If both players decide to bet, the cards are revealed and the player with the higher card wins all the chips currently in game. If either of players decides to fold, he immediately loses and all the chips currently in game are won by his opponent. If the players have cards of the same value, the result is a tie and the chips are split evenly between the players. Write all the variables and what they correspond to in sequence-form LP, that has a player's 2 part of Nash equilibrium as a solution.

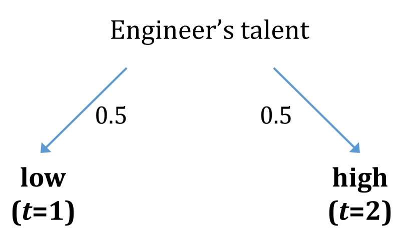
Solution:

1. Realization plan $r(\sigma)$ is probability that sequence σ is played. Behavioral strategy is a probability of playing an action a in infoset I. For sequence σ that leads to I, the behavioral strategy is extracted as $\pi(I,a) = \frac{r(\sigma a)}{r(\sigma)}$.
2. The least amount of infosets is when the only information players have is the action they played. This means that each preceeding infoset create A new infosets. So we get a following formula $1 + A + A^2 + \cdots + A^{d-1} = \sum_{i=0}^{d-1} A^i$ for each player, resulting in total of $2\sum_{i=0}^{d-1} A^i$. The maximal amount of infosets is in perfect information game, so the amount of infosets is equal to the amount of histories: $1 + A + A^2 + \cdots + A^{2d-1} = \sum_{i=0}^{2d-1} A^i$ infosets.
3. Variable r for each sequence, so in this case it is $r(\emptyset), r(Fold_J), r(Fold_Q), r(Fold_K), r(Fold_A), r(Bet_J), r(Bet_Q), r(Bet_K), r(Bet_A)$ and also a variable for each opponent's infoset $v(I_J), v(I_Q), v(I_K), v(I_A)$

3. (5 pts) An Engineer has either a low ($t = 1$) or a high ($t = 2$) talent with equal probability ($p = 1/2$). The value of the Engineer's talent is private information to the Engineer. The Engineer's pure strategies are applying for a job or being an entrepreneur and doing a startup. The Company's pure strategies are either hiring or not hiring the Engineer.

   If the Engineer applies for the job and but the Company does not hire him/her, then the Engineer becomes an entrepreneur and does a startup. The utility of the Engineer is t (talent) from being an entrepreneur, and w (wage) from being hired for a job. The utility of the Company is $t - w$ if it hires the Engineer for a job, and 0 otherwise.

   See the following payoff matrices for details.



Low ($t=1$) — *Company* (columns), *Engineer* (rows):

	Hire	Not Hire
Startup	1,0	1,0
Job	$w, 1-w$	1, 0

High ($t=2$) — *Company* (columns), *Engineer* (rows):

	Hire	Not Hire
Startup	2,0	2,0
Job	$w, 2-w$	2, 0

(a) What is the best strategy for the Engineer and how does it depend on his/her talent and on the value of w?

(b) What is the best strategy for the Company and how does it depend on the value of w?

(c) For which value(s) of w does there exist a mixed Bayes-Nash equilibirum, which is *not* a pure equilibirum?

Solution:

a)

t=1 (low):

- $w < 1$: startup
- $w = 1$: any mixture
- $w > 1$: job

t=2 (high):

- $w < 2$: startup
- $w = 2$: any mixture
- $w > 2$: job

b)

- $w < 1$: *hire*
- $w = 1$: any mixture
- $w > 1$: *not hire* (expected payoff of *hire* is $0.5 * (1 - w) < 0$; expected payoff of *not hire* is zero).
  NOTE: It is is necessary to consider the dominant strategy for each type of the Engineer player, consider the Company's best response and only then take the expectation over the distribution of Engineer types. E.g. if $w = 1.5$, then the low talent engineer would ALWAYS apply to the job, while the high talent engineer would NEVER apply for the job (*startup* is dominant strategy for him/her), i.e., the Company would be making the expected loss of $0.5 * (1 - 1.5) = -0.25$ on expectation.

c)

- For $w \leq 1$ because both low and high talent engineers would choose *startup* and any mixture of the Company's actions is the best response)
- For $w > 1$ because in this case the Company would choose *not hire* and any mixture of Engineer's actions (for both talents) is the best respose.

4. (5 pts) Consider a combinatorial Vickerey-Clarke-Groves (VCG) auction with three bidders $\{a, b, c\}$, two goods $\{g_1, g_2\}$ and the following valuations of bundles:

   1. What would be the bids the agents would place in this auctions?
   2. What would be the allocation of goods resulting from the auction?
   3. What would be the payments the agents would make in the auction?

bundle	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_a	2	1	6
v_b	2	4	4
v_c	3	2	5

4. Would the allocation of the goods be *efficient*? Justify your answer.
5. Would the outcome of the auction be *revenue maximizing*? Justify your answer.

Solution:

a) the agents would bid their true valuations (for all bundles of goods)
b) bidder a gets nothing; bidder b gets g_2, bidder c gets g_3
c) bidder a pays nothing; bidder b pays 3; bidder c pays 2
d) yes, VCG is efficient and the goods are allocated in a way maximizing social wellfare (i.e. the sum of utilities bidders receive from the allocation)
e) no, the VCG is not revenue maximizing and there is potential to increase the expected revenue by considering other types of actions (e.g. with a reserve price)

5. Solve the following problems and always explain your answers.
   (a) (2 pts) Does every weighted voting game have a nonempty core?
   (b) (2 pts) Is the Banzhaf index of a non-veto player always nonzero?
   (c) (3 pts) Find a 5-player weighted voting game with precisely 1 vetoer and 1 null player.
   (d) (3 pts) Let v be a 3-player coalitional game given by

$$v(1) = v(2) = v(13) = 1, v(3) = 0, v(12) = v(23) = 2, v(123) = 3.$$

   Show that the core of v is a line segment in $\mathbb{R}^3$.

Solution:

(a) No. The counterexample is pure majority voting with $n = 3$ players, quota $q = 2$, and weights $(1, 1, 1)$.

(b) No, since a null-player is never the vetoer and its Banzhaf index is zero.

(c) This is, for example, the game with weights $(3, 1, 1, 1, 0)$ and quota 4.

(d) Game v is supermodular, so its core is the convex hull of the marginal vectors. There are only two different marginal vectors, $(1, 1, 1)$ and $(1, 2, 0)$.