Surname and name: ____________________

Task	1	2	3	4	5	*Total*
Maximum	10	10	4	6	10	40
Points						

1. (a) (2 pts) Is the following assertion true or not? "Every 2-player strategic game with strategy spaces of size 2 each has a Nash equilibrium in which one player uses a pure strategy."

   (b) (3 pts) Formulate a linear program to compute the equilibrium strategy of the maximizing (row) player in the matrix game
$$\begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}.$$

   (c) (5 pts) Formulate an optimization problem to maximize the social welfare of correlated equilibria in a two-player game in which the payoffs are given by
$$\begin{bmatrix} 1,1 & 2,3 \\ 0,1 & 3,2 \end{bmatrix}$$
   and decide if the problem has an optimal solution.

Solution:

(a) False. The counterexample is Matching Pennies. (b) Maximize v subject to $1 \geq p \geq 0$, and $2p + (1-p) \geq v$, $-4p + 3(1-p) \geq v$ where $v \in \mathbb{R}$. (c) Maximize
$$\sum_{i=1,2} \sum_{s \in \{1,2\}^2} p(s) u_i(s)$$
subject to $p(s) \geq 0$, for all $s \in \{1,2\}^2$, $\sum_{s \in \{1,2\}^2} p(s) = 1$, and
$$\begin{aligned} 0p(1,1) + 3p(1,2) &\leq 1p(1,1) + 2p(1,2) \\ 1p(2,1) + 2p(2,2) &\leq 0p(2,1) + 3p(2,2) \\ 2p(1,1) + 3p(2,1) &\leq 1p(1,1) + 1p(2,1) \\ 1p(1,2) + 1p(2,2) &\leq 3p(1,2) + 2p(2,2) \end{aligned}$$

2. (a) (2 pts) Consider a two-player zero-sum game with perfect recall, where both players have I infosets and A actions in each infoset. What is the number of variables and constraints of the sequence-form linear program for computing the first player's part of Nash equilibrium.

   (b) (3 pts) Can any extensive-form game be transformed into a normal-form game? Either write down such a transformation or find an example of an extensive-form game that cannot be transformed to a normal-form.

(c) (5 pts) Consider the following two-player card game. The game begins with a mandatory bet of 1 chip for both players. After the bet, both players obtain 1 card randomly chosen from the deck of 5 cards containing two Js, two Qs, and one K. The players know their own card but do not observe the card of the opponent. After the cards are dealt, player 1 decides if he wants to continue playing by betting another chip or end the game by folding. If he decides to bet, player 2 also decides between betting 1 chip or folding. If both players decide to bet, the cards are revealed and the player with the higher card wins all the chips currently in game. If either of players decides to fold, he immediately loses and all the chips currently in game are won by his opponent. If the players have cards of the same value, the result is a tie and the chips are split evenly between the players. Write down all the sequences and pure strategies in this game for player 2.

Solution:

1. Variables: For each sequence a single variable. This corresponds to each action in each infoset and empty sequence $IA + 1$. For each opponents infoset a single variable I. Could be $I + 1$ if artificial root is constructed. Totally: $IA + I + 1$.
   Constraints: For each opponent's infoset and each action in this infoset there is a constraint IA. If the artificial root is constructed it is $IA + 1$ For each sequence of player there is a constraint that realization plan is greater then 0 IA. Lastly for each infoset of player, it has to hold, that realization plan that leads to this infoset is the same as sum of realization plans that leads out of this sinfoset I. Totally: $IA+IA+I$
2. Yes, the procedure starts with listing all the pure strategies in the game. For each combination of pure strategies of each player, the utility is given by the weighted sum of all the reached terminal nodes (weighted by a chance node).
3. $\Sigma_2 = \{\emptyset, Fold_J, Fold_Q, Fold_K, Bet_J, Bet_Q, Bet_K\}$
   $\mathcal{S}_2 = \{Fold_J Fold_Q Fold_K, Fold_J Fold_Q Bet_K, Fold_J Bet_Q Fold_K, Fold_J Bet_Q Bet_K,$
   $Bet_J Fold_Q Fold_K, Bet_J Fold_Q Bet_K, Bet_J Bet_Q Fold_K, Bet_J Bet_Q Bet_K, \}$

3. Assuming bidders are risk neutral and have independent private valuations drawn from a common cumulative distribution that is strictly increasing on $[0, 1]$:

   (a) (2 pts) Which of the following auction mechanisms are *strategically equivalent*: i) Dutch, ii) First-price sealed bid, iii) Second-price sealed bid, iv) Japanese, v) English?

   (b) (2 pts) Which of the following auction mechanisms are *revenue equivalent*? i) Dutch, ii) First-price sealed bid, iii) Second-price sealed bid, iv) Second-price sealed bid with a reserve price, v) English, and vi) Japanese.

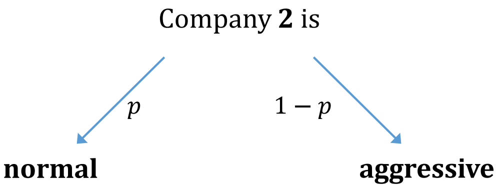
Solution:

(a) Dutch and First-price sealed bid are strategically equivalent. Second-price sealed bid, Japanese and English are strategically equivalent.

(b) All are revenue equivalent except for the Second-price sealed bid with a reserve price (which produces higher expected revenue).

4. (6 pts) Company 1 is choosing whether to enter a market or stay out. Company 2 is already in the market and chooses (simultaneously) whether to fight. Company 2 can either be a *normal* company (with probability p) or an *aggressive* company (with probability $1 - p$). The payoffs of Company 2 depend on its type (which is known to it).

   See the following payoff matrices for details.



		Company ***2***	
		Fight	Not Fight
Company ***1***	Enter	$-1,0$	$1,2$
	Out	$0,3$	$0, 3$

		Company ***2***	
		Fight	Not Fight
Company ***1***	Enter	$-1,2$	$1, -2$
	Out	$0,3$	$0, 3$

(a) What is the best strategy for Company 2 and how does it depend on p (if at all)?
(b) What is the best strategy for Company 1 and how does it depend on p (if at all)?
(c) What is the expected payoff for Company 1 when playing its best strategy and how does the payoff depend on p (if at all)?
(d) What is the pure Bayes-Nash equilibrium in this game? Does it depend on p? How?

Solution:

(a) *Not Fight* if of type normal. *Fight* if aggressive. The best strategy only depends on the type of Company 2 and not on p.

(b) The best strategy depends on the value of p. If $p > \frac{1}{2}$, then the best strategy is to *Enter*. If $p < \frac{1}{2}$ then the best strategy is to *Stay out*. If $p = \frac{1}{2}$, then any mixture is the best strategy.

(c) The expected pay-off depends on the value of p. If $p > \frac{1}{2}$, then the expected payoff is $2p - 1$. If $p \leq \frac{1}{2}$, then the expected payoff is 0.

(d) Bayes-Nash equilibirum depends on p:

- $p > \frac{1}{2}$: Company 1 should *Enter*; Company 2 should *Not Fight* if of type normal and *Fight* if of type aggressive.
- $p < \frac{1}{2}$: Company 1 should *Stay Out*; Company 2 the same as above.
- $p = \frac{1}{2}$: Company 1 any mixture; Company 2 the same as above.

5. Consider a simple game with 4 players in which the winning coalitions are $123, 124, 234, 1234$.
   - (a) (1 pts) Is this a weighted voting game?
   - (b) (2 pts) Which players are vetoers, dictators, and null players?
   - (c) (3 pts) Compute the Banzhaf index for each player.
   - (d) (2 pts) Is the core of this game nonempty? Decide and explain without computation.
   - (e) (2 pts) Check if the game is supermodular or superadditive.

Solution:

(a) This is a weighted voting game with weights $(1, 2, 1, 1)$ and quota 4. (b) The only vetoer in the game is 2. There are neither dictators (no single player coalition is winning) nor null players (every player makes some loosing two-player coalition winning). (c) The vector of Banzhaf indices is $(\frac{2}{8}, \frac{4}{8}, \frac{2}{8}, \frac{2}{8})$. (d) The game has nonempty core since the core allocation is $(0, 1, 0, 0)$. (e) The game is not supermodular since

$$2 = v(123) + v(124) > v(1234) + v(12) = 1.$$

The game is superadditive since there is no pair of disjoint winning coalitions.