Assignment 1 (Bernoulli VAE). Let us consider a VAE with binary valued latent variables \( z \in \mathcal{Z} = \{0, 1\}^n \). Training such VAEs by maximising the ELBO criterion requires computation of the gradient of the data term w.r.t. encoder parameters \( \varphi \)

\[
\nabla_\varphi \mathbb{E}_{q_\varphi(z \mid x)} \log p_\theta(x \mid z) = \nabla_\varphi \sum_{z \in \mathcal{Z}} q_\varphi(z \mid x) \log p_\theta(x \mid z).
\]  

(1)

Here we can not apply the re-parametrisation trick as in the case of Gaussian latent variables.

a) We can explicitly sum over \( z \in \mathcal{Z} \) if the dimension of the latent space is small. This is however not possible for high dimensional latent spaces.

b) (Score function, log-trick) Prove the following equality

\[
\nabla_\varphi \sum_{z \in \mathcal{Z}} q_\varphi(z \mid x) \log p_\theta(x \mid z) = \sum_{z \in \mathcal{Z}} q_\varphi(z \mid x) \nabla_\varphi \log q_\varphi(z \mid x) + \nabla_\varphi \log q_\varphi(z \mid x)
\]

Conclude that the following procedure implements an unbiased stochastic estimator of the required gradient (1):

Sample \( z \sim q_\varphi(z \mid x) \) and compute \( \log p_\theta(x \mid z) \nabla_\varphi \log q_\varphi(z \mid x) \)

Assignment 2 (VAE: Sticked Landing). When learning VAEs by maximising the ELBO criterion, we must compute the gradient of the KL-divergence term, i.e.

\[
\nabla_\varphi \sum_{z \in \mathcal{Z}} q_\varphi(z \mid x) \left[ \log q_\varphi(z \mid x) - \log p(z) \right],
\]

where \( p(z) \) denotes the prior distribution on the latent space.

a) Usually, this KL-divergence can be computed in closed form. In case of a Bernoulli VAE discussed in the previous assignment, this amounts to compute the KL-divergence for pairs of Bernoulli distributions. Give a formula it.

b) Let us now consider the gradient of the first term in the formula above.

\[
\nabla_\varphi \sum_{z \in \mathcal{Z}} q_\varphi(z \mid x) \log q_\varphi(z \mid x) = \sum_{z \in \mathcal{Z}} \log q_\varphi(z \mid x) \nabla_\varphi q_\varphi(z \mid x) + \sum_{z \in \mathcal{Z}} q_\varphi(z \mid x) \nabla_\varphi \log q_\varphi(z \mid x)
\]

Prove that the second sum is always zero.
Assignment 3 (Score function). The score function approach discussed in the first assignment provides an unbiased gradient estimator. However, its key drawback is its high variance. Let us study this on a simple example. Consider the derivative

$$\frac{d}{d\beta} \mathbb{E}_\beta[z] = \frac{d}{d\beta} \sum_z q_\beta(z) z,$$

where \(q_\beta(z) = \beta^z (1 - \beta)^{(1-z)}\) is a Bernoulli distribution for a binary variable \(z = 0, 1\).

a) Show that this derivative equals 1.

b) Let us now consider the score function approach for this derivative.

$$\frac{d}{d\beta} \mathbb{E}_\beta[z] = \sum_z q_\beta(z) z \frac{d}{d\beta} \log q_\beta(z)$$

Show that \(z \frac{d}{d\beta} \log q_\beta(z) = \frac{z}{\beta}\). Compute the variance of estimating this random variable on an i.i.d. sample \(\{z_i | i = 1, \ldots, m\}\) generated from \(q_\beta\). How is it depending on \(\beta\)?