VI extensions

6. května 2019

B4M36PUI/BE4M36PUI — Planning for Artificial Intelligence

Outline

- Review of MDP concepts
- Value Iteration algorithm
- VI extensions

Review of last tutorial

Value function of a policy

Look at the following definition of a value function of a policy for inifnite-horizon MDP. It contains multiple mistakes, correct them on a piece of paper:

Def: Value function of a policy for infinite-horizon MDP

Assume infinite horizon MDP with $\gamma \in [0, 100]$. Then let Value function of a policy π for every state $s \in S$ be defined as

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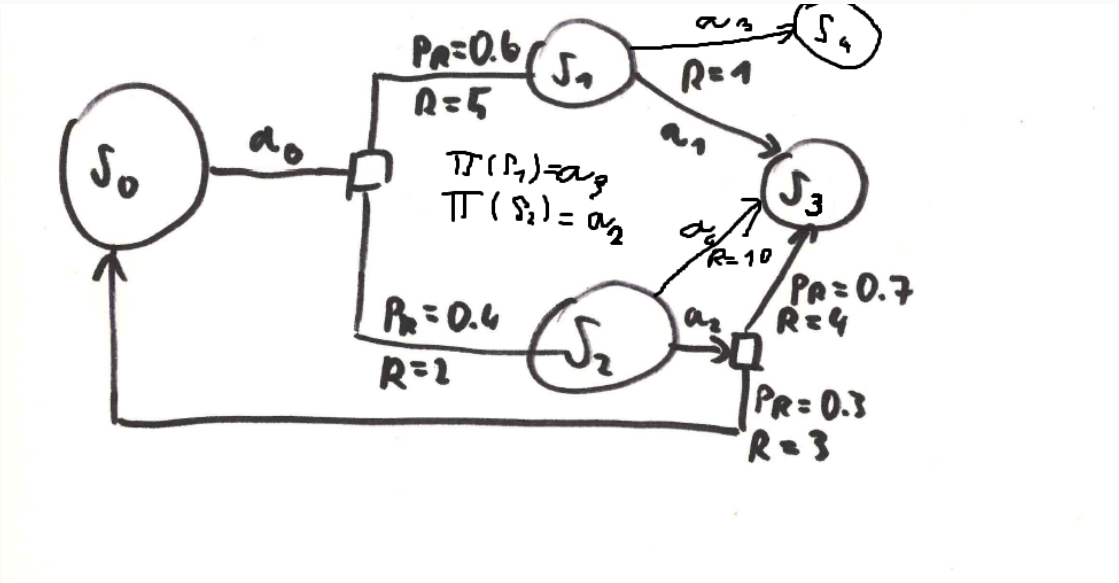
Question: Difference to def. of an optimal value function?

Bellman Equations

Write down equations for finding a value function of a policy π. How would you solve these equations?

- T : $T(S_0, a_0, S_1) = 0.6$
  $T(S_0, a_0, S_2) = 0.4$
  $T(S_1, a_1, S_3) = 1$
  $T(S_2, a_2, S_3) = 0.7$
  $T(S_2, a_2, S_0) = 0.3$
- R : $R(S_0, a_0, S_1) = 5$
  $R(S_0, a_0, S_2) = 2$
  $R(S_1, a_1, S_3) = 1$
  $R(S_2, a_2, S_3) = 4$
  $R(S_2, a_2, S_0) = 3$
  $R(S_1, a_3, S_4) = 4$
  $R(S_2, a_4, S_3) = 10$

- S: S_0, S_1, S_2, S_3, S_4
- A: a_0, a_1, a_2, a_3, a_4)



Value Iteration

value Iteration algorithm

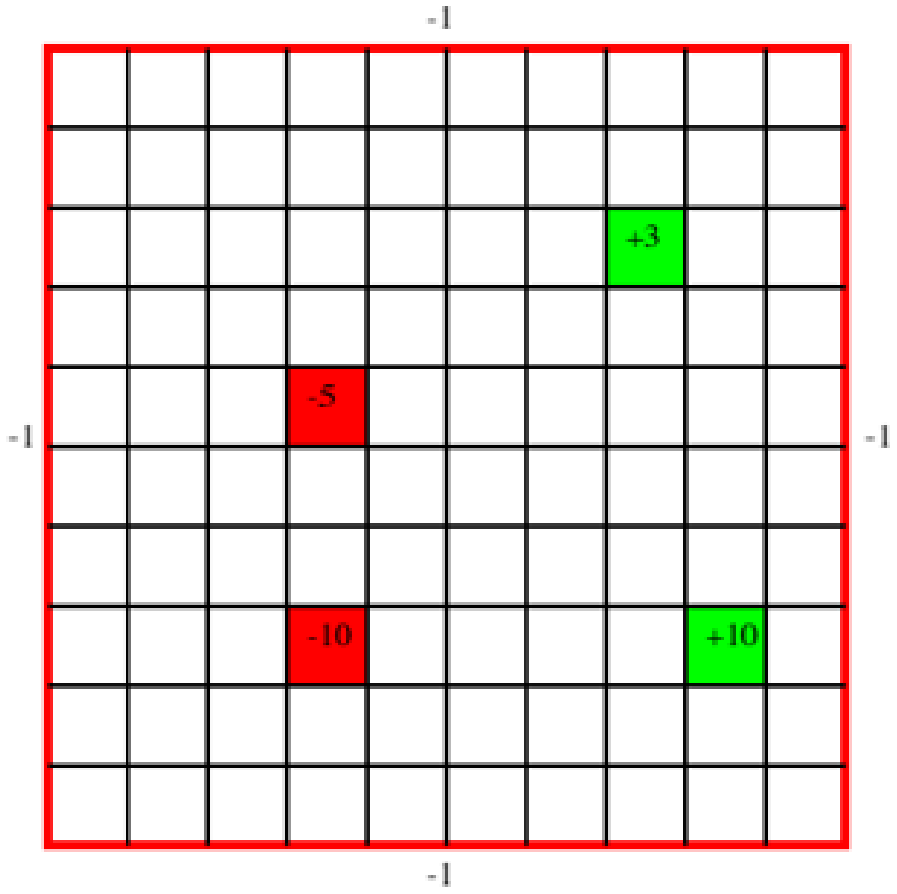
Basic algorithm for finding solution of Bellman Equations iteratively.

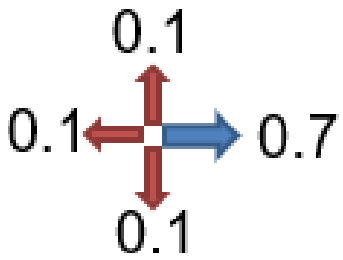
1. initialize V_0 arbitrarily for each state, e.g to 0, set $n = 0$
2. Set $n = n + 1$.
3. Compute Bellman Backup, i.e. for each $s \in S$:
   3.1 $V_n(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma V_{n-1}(s')]$
4. GOTO 2.

VI example

Robot Emil-like domain.
Assuming $\gamma = 1$.
Using VI, initializing
$\forall s \; V_0(s) = 0$, calculate
V_1, V_2, V_3 for the nine states
around $+10$ tile.

- Moving into edges gives -1 reward
- Moving onto marked tiles gives corresponding reward





From http://artint.info/html/ArtInt_224.html#gridworld-ex

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Question: Does it converge? How fast? When do we stop?

Residual - definition

Def: Residual

Residual of value function V_n from V_{n+1} at state $s \in S$ is defined by:

$$Res^{V_n}(s) = |V_n(s) - V_{n+1}(s)|$$

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Residual of value function V from V' is given by:

$$Res^{V_n} = ||V_n - V_{n+1}||_\infty = \max_s |V_n(s) - V_{n+1}(s)|$$

VI stopping criterion

Stopping criterion: When residual of consecutive value functions is below low value of ϵ:

$$||V_n - V_{n+1}|| < \epsilon$$

However, this does not imply ϵ distance of value of greedy policy from optimal value function.

👁 for proof, see: [Singh, Yee: An Upper Bound on the Loss from Approximate Optimal-Value Functions, p.229]

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Theorems for general MDP exist of form:

$$V_n, V^* \text{as above} \implies \forall s \; |V_n(s) - V^*(s)| < \epsilon(\text{Some MDP dependent term})$$

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In case of discounted ($\gamma < 1$) infinite-horizon MDPs:

$$V_n, V^* \text{as above} \implies \forall s \; |V_n(s) - V^*(s)| < 2\frac{\epsilon\gamma}{1-\gamma}$$

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VI with stopping criterion

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Question: What is the policy?

- *Greedy policy* π_n^V is the policy given as argmax of V_n.

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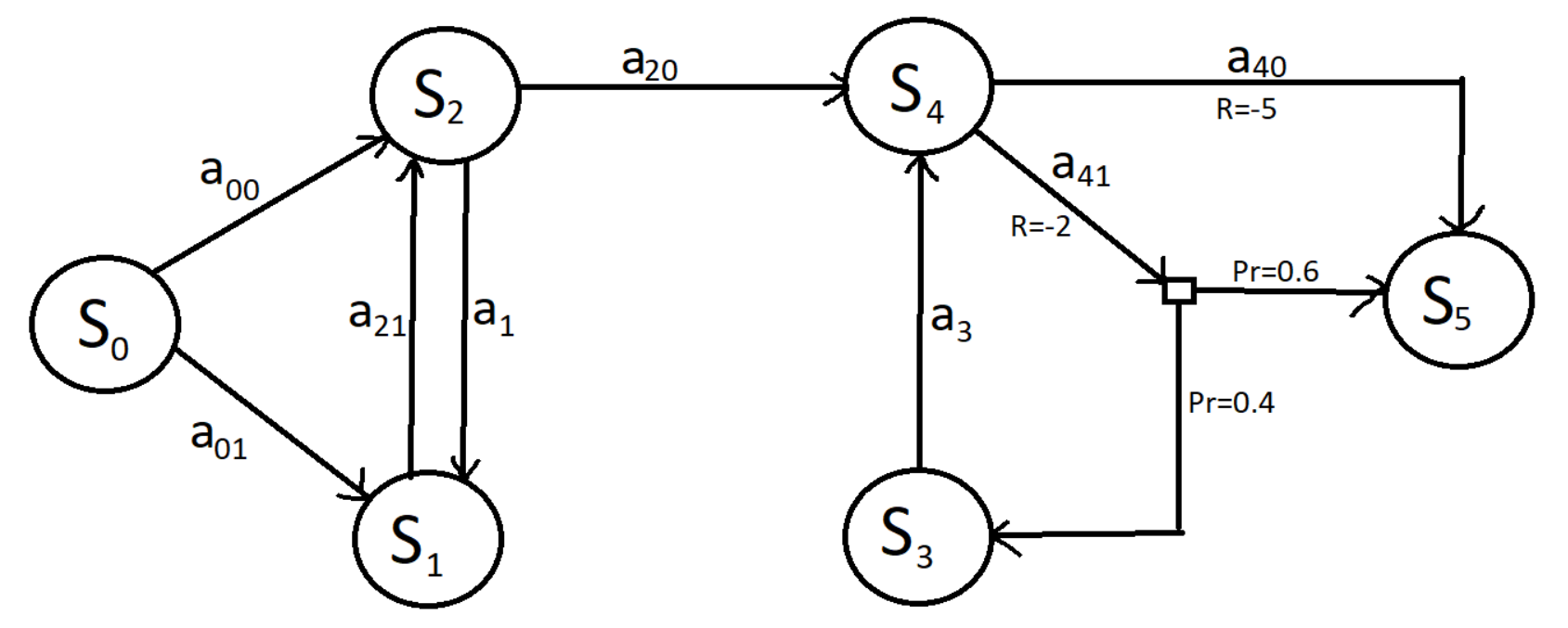
Question: What are the memory requirements of VI?

- Value of each state needs to be stored twice

VI extensions

Another beautiful MDP example



- All undeclared rewards are -1

Task: Initialize VI with negative distance to S_5 and calculate first 3 iterations of VI, with state ordering S_0 to S_5

Asynchronous VI

1. initialize V_0 arbitrarily for each state, e.g to 0
2. While $Res^V > \epsilon$, do:
   - 2.1 pick some state s
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Question: How to pick s in 2.1?

- Simplest is *Gauss-Seidel VI*, that is run AVI over all states iteratively

Prioritized VI

How else can we pick states to update? (Ans: by ordering them in clever ways)

- Build priority queue of states to update $\rightarrow$ *Prioritized Sweeping VI*. Update states in the order of the queue. Priority function:

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EXAMPLE ON BOARD

Convergence?

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- OR If you interleave regular VI sweeps with Prioritized VI

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Subdivide state space into partitions. Pre-solve parts independently.

- Example is *Topological VI*

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