Outline

- Scope of MAS in Game Theory
- Game representations
  - Normal-form games
- What are the problems compared to, e.g., planning
- Analysis of a game
- Properties and computation of Nash equilibrium
- Game modeling
Noncooperative Game Theory

- Single round games
  - Normal-form games
  - Extensive-form games
  - MAIDS, Congestion games

- Multiple round games
  - Repeated games
  - Stochastic games
Types of games

- Two-player vs n-player
- Zero-sum games vs general-sum games
- Sequential vs one-shot
- Perfect-information vs imperfect-information
- Finite vs infinite
Types of games

- Two-player vs n-player
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Normal-form games

- Players set $\mathcal{N} = \{1, \ldots, n\}$
- Actions set $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$
- Utility functions $u = \langle u_1, \ldots u_n \rangle$, where $u_i : \mathcal{A} \rightarrow \mathbb{R}$
Normal-form games

- Represented as n-dimensional matrix
- Every entry is n-dimensional tuple of utilities for every player
A pure strategy $a_i$ in normal-form games represents the choice of specific action $a_i \in \mathcal{A}_i$ for player $i$

A mixed strategy $s_i$ is a strategy distribution over pure strategies

Strategy profile $a/s$ is a set of pure/mixed strategies, one for every player

Overloading of utility function $u(a_i, a_{-i})$, $u(s_i, s_{-i})$, $u(s)$
Why do we need Game Theory?
Approaches for reasoning about games

- Studying game structure/properties
  - Social welfare optimality
  - Pareto optimality

- Stable strategies (solution concepts)
  - Maxmin
  - Minmax
  - Nash equilibrium
  - Stackelberg equilibrium
  - Correlated equilibrium

- Computation helpers
  - Dominance
Social welfare

- Defined as
  \[ WF = \sum_{i \in N} u_i(s) \]  
  (1)

- Not stable against deviations
- Cooperative players
Pareto optimality

- Reasoning about outcomes
- Outcome $o$ pareto dominates outcome $o'$ iff
  \[
  \forall i \in \mathcal{N} \ o_i \geq o'_i \text{ and } \exists i \in \mathcal{N} \ o_i > o'_i
  \]
  (2)
- Outcome $o$ is pareto optimal if it is not pareto dominated by any other outcome $o'$
Dominance

- **Strict dominance**
  - Strategy $a_i$ strictly dominates $a'_i$ iff
    $$\forall a_{-i} \in A_{-i} : u_i(a_i, a_{-i}) > u_i(a'_i, a_{-i})$$
    (3)

- **Weak dominance**
  - Strategy $a_i$ weakly dominates $a'_i$ iff
    $$\forall a_{-i} \in A_{-i} : u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ and}$$
    $$\exists a_{-i} \in A_{-i} : u_i(a_i, a_{-i}) > u(a'_i, a_{-i})$$
    (4)

- **Very weak dominance**
  - Strategy $a_i$ very weakly dominates $a'_i$ iff
    $$\forall a_{-i} \in A_{-i} : u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$$
    (6)
Nash equilibrium

- A strategy $s_i^*$ is the best response to strategies $s_{-i}$, written as $s_i^* \in BR(s_{-i})$ iff

$$\forall s_i \in S_i \ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad (7)$$

- Nash equilibrium
  - Strategy profile $s = \{s_1, ..., s_n\}$ is a Nash equilibrium iff

$$\forall i \in \mathcal{N} \ s_i \in BR(s_{-i}) \quad (8)$$

- Stable against deviations of players as every player plays his best response to the strategies of the rest
- Assumes self-interested rational players
- Every finite game has a non-empty set of Nash equilibria
- Examples
Properties of NE

- Values in NE might differ
- Strategies not interchangeable
- Mistake of the opponent might hurt me
Properties of NE in zero-sum games

- All NE have the same value for $i$ (value of the game)
- The value is guaranteed (mistakes of the opponent only increase my expected outcome)
- Strategies are interchangeable between NE
- $\text{minmax} = \text{maxmin} = \text{NE} = \text{SE}$