Solving Normal-Form Games

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(based on slides of Branislav Bošanský)

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October 30, 2018
Previously ... on multi-agent systems.
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... and now we continue ...
Rock Paper Scissors

- 2 players...
Rock Paper Scissors

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- What are the actions of the players? \((A_1, A_2)\)
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- What are the possible outcomes?
What is the best strategy to play in Rock-Paper-Scissors?

Every time we are about to play we randomly select an action we are going to use.

The concept of pure strategies is not sufficient.
Rock Paper Scissors

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Mixed Strategies

Definition (Mixed Strategies)

Let $G = (N, A, u)$ be a normal-form game. Then the set of mixed strategies $S_i$ for player $i$ is the set of all probability distributions over $A_i$;

$$S_i = \Delta(A_i)$$

Player selects a pure strategy according to the probability distribution. We use $S_{-i}$ to denote strategies of all other players except player $i$.

We extend the utility function to correspond to expected utility:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

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Let \( G = (N, A, u) \) be a normal-form game. We say that \( s_i \) strongly dominates \( s_i' \) if 
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\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s_{i}', s_{-i}).
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Best Response and Equilibria

Definition (Best Response)

Let $G = (N, A, u)$ be a normal-form game and let $\text{BR}_i(s - i) \subseteq S_i$ such that $s^*_i \in \text{BR}_i(s - i)$ if and only if $\forall s_i \in S_i, u_i(s^*_i, s - i) \geq u_i(s_i, s - i)$.

Definition (Nash Equilibrium)

Let $G = (N, A, u)$ be a normal-form game. Strategy profile $s = \langle s_1, ..., s_n \rangle$ is a Nash equilibrium iff $\forall i \in N, s_i \in \text{BR}_i(s - i)$.
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Theorem (Nash)

Every game with a finite number of players and action profiles has at least one Nash equilibrium in mixed strategies.
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Support of Nash Equilibria

Definition (Support)
The support of a mixed strategy $s_i$ for a player $i$ is the set of pure strategies $\text{Supp}(s_i) = \{a_i | s_i(a_i) > 0\}$.

Question
Assume Nash equilibrium $(s_i, s_{-i})$ and let $a_i \in \text{Supp}(s_i)$ be an (arbitrary) pure strategy from the support of $s_i$. Which of the following possibilities can hold?

- $u_i(a_i, s_{-i}) < u_i(s_i, s_{-i})$
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Let $s \in S$ be a Nash equilibrium and $a_i, a'_i \in A_i$ are actions from the support of $s_i$. Now, $u_i(a_i, s_{-i}) = u_i(a'_i, s_{-i})$. 
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Can we exploit this fact to find a Nash equilibrium?
Finding Nash Equilibria

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Column player (player 2) plays $L$ with probability $p$ and $R$ with probability $(1 - p)$. In NE it holds $E_{u_1}(U) = E_{u_1}(D)$.

Similarly, we can compute the strategy for player 1 arriving at $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$ as Nash equilibrium.
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$$2p + 0(1 - p) = 0p + 1(1 - p)$$

$$p = \frac{1}{3}$$
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Not really... No strategy $s_i$ of the row player ensures

$$u_{-i}(s_i, L) = u_{-i}(s_i, C) = u_{-i}(s_i, R) :-( $$
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Iterated removal of dominated strategies...
Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem. Playing a Nash strategy does not give any guarantees for the expected payoff. If we want guarantees, we can use a different concept – maxmin strategies.

**Definition (Maxmin)**

- The **maxmin strategy** for player $i$ is $\arg \max s_i \min s_{-i} u_i(s_i, s_{-i})$.
- The **maxmin value** for player $i$ is $\max s_i \min s_{-i} u_i(s_i, s_{-i})$.

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The *maxmin strategy* for player $i$ is $\arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player $i$ is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$. 

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Maxmin and Minmax

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Minmax strategies represent punishment strategies for player $-i$. Maxmin strategies are conservative strategies against a worst-case opponent.
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Definition (Minmax, two-player)

In a two-player game, the minmax strategy for player $i$ against player $-i$ is $\arg \min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the minmax value for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$. 
Maxmin and Minmax

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Maxmin strategies are conservative strategies against a worst-case opponent.

Minmax strategies represent punishment strategies for player \(-i\).
What is the maxmin strategy for the row player in this game?

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Zero-sum case

What about zero-sum case? How do
- player $i$’s maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
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\[
\begin{align*}
\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) &= - \min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})
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- player $i$'s maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
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relate?

\[
\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = -\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})
\]

... but we can prove something stronger ...
Maxmin and Von Neumann’s Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.
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2. we can safely play Nash strategies in zero-sum games
3. all Nash equilibria have the same payoff (by convention, the maxmin value for player 1 is called value of the game).
Computing NE in Zero-Sum Games

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

\[
\max_{s,U} \quad U(s)
\]

\[
\text{s.t.} \quad \sum_{a_1 \in A_1} s(a_1) u_1(a_1, a_2) \geq U \quad \forall a_2 \in A_2
\]

\[
\sum_{a_1 \in A_1} s(a_1) = 1
\]

\[
s(a_1) \geq 0 \quad \forall a_1 \in A_1
\]

Computing a Nash equilibrium in zero-sum normal-form games can be done in polynomial time.
Computing NE in Zero-Sum Games

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\[
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\max & \quad \mathbf{u} \in \mathbb{R}^n \\
\text{s.t.} & \quad \sum_{\mathbf{a}_1 \in \mathcal{A}_1} \mathbf{s}(\mathbf{a}_1) \mathbf{u}(\mathbf{a}_1, \mathbf{a}_2) \geq \mathbf{u} \quad \forall \mathbf{a}_2 \in \mathcal{A}_2 \\
& \quad \sum_{\mathbf{a}_1 \in \mathcal{A}_1} \mathbf{s}(\mathbf{a}_1) = 1 \\
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$$\max_{s,U} U$$

subject to

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(2)

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(4)
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Computing NE in General-Sum Games

The problem is more complex for general-sum games (LCP program):

\[
\sum_{a_2 \in A_2} u_1(a_1, a_2) s_2(a_2) + q(a_1) = U_1 \forall a_1 \in A_1
\]

\[
\sum_{a_1 \in A_1} u_2(a_1, a_2) s_1(a_1) + w(a_2) = U_2 \forall a_2 \in A_2
\]

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\[q(a_1) \geq 0, \; w(a_2) \geq 0, \; s_1(a_1) \geq 0, \; s_2(a_2) \geq 0 \forall a_1 \in A_1, \forall a_2 \in A_2
\]

\[
s_1(a_1) \cdot q(a_1) = 0, \; s_2(a_2) \cdot w(a_2) = 0 \forall a_1 \in A_1, \forall a_2 \in A_2
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Computing a Nash equilibrium in two-player general-sum normal-form game is a PPAD-complete problem. The problem gets even more complex (FIXP-hard) when moving to \(n \geq 3\) players.
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\[q(a_1) \geq 0, \ w(a_2) \geq 0, \ s_1(a_1) \geq 0, \ s_2(a_2) \geq 0 \quad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2\]

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Computing a Nash equilibrium in two-player general-sum normal-form game is a PPAD-complete problem. The problem gets even more complex (FIXP-hard) when moving to \( n \geq 3 \) players.
Regret

The concept of regret is useful when the other players are not completely malicious.

\[(1, a)\] \[\epsilon\] \[(2, c)\] \[(1, d)\]

Definition (Regret)

A player's regret for playing an action \(a_i\) if the other agents adopt action profile \(a_{-i}\) is defined as

\[
\max_{a_i' \in A_i} u_i(a_i', a_{-i}) - u_i(a_i, a_{-i})
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Regret

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**Definition (Regret)**

A player $i$’s *regret* for playing an action $a_i$ if the other agents adopt action profile $a_{−i}$ is defined as

$$\max_{a_i' \in A_i} u_i(a'_i, a_{−i}) - u_i(a_i, a_{−i})$$
Regret

Definition (MaxRegret)
A player $i$'s maximum regret for playing an action $a_i$ is defined as $\max_{a \in A} - (\max_{a' \in A} u(a', a - i) - u(a_i, a - i))$.

Definition (MinimaxRegret)
Minimax regret actions for player $i$ are defined as $\arg \min_{a_i \in A_i} \max_{a - i \in A - i} (\max_{a' \in A} u(a', a - i) - u(a_i, a - i))$. 

Definition (MaxRegret)

A player is maximum regret for playing an action $a_i$ is defined as

$$
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Definition (MaxRegret)

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Consider again the following game:

\begin{tabular}{|c|c|c|}
\hline
 & L & R \\
\hline
U & (2, 1) & (0, 0) \\
D & (0, 0) & (1, 2) \\
\hline
\end{tabular}

Wouldn't it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination?

We can use a correlation device—a coin, a streetlight, commonly observed signal—and use this signal to avoid unwanted outcomes.
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Wouldn’t it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination? We can use a correlation device—a coin, a streetlight, commonly observed signal—and use this signal to avoid unwanted outcomes.
Correlated Equilibrium

Definition (Correlated Equilibrium (simplified))

Let \( G = (N, A, u) \) be a normal-form game and let \( \sigma \) be a probability distribution over joint pure strategy profiles \( \sigma \in \Delta(A) \).

We say that \( \sigma \) is a correlated equilibrium if for every player \( i \), every signal \( a_i \in A_i \) and every possible action \( a'_i \in A_i \) it holds

\[
\sum_{a_j \in A_j} \sigma(a_i, a_j) u_i(a_i, a_j) \geq \sum_{a_j \in A_j} \sigma(a_i, a_j) u_i(a'_i, a_j)
\]

Corollary

For every Nash equilibrium there exists a corresponding Correlated Equilibrium.
Correlated Equilibrium

Definition (Correlated Equilibrium (simplified))

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$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i', a_{-i})$$
Correlated Equilibrium

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Let $G = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $\sigma$ be a probability distribution over joint pure strategy profiles $\sigma \in \Delta(\mathcal{A})$. We say that $\sigma$ is a correlated equilibrium if for every player $i$, every signal $a_i \in \mathcal{A}_i$ and every possible action $a'_i \in \mathcal{A}_i$ it holds

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**Corollary**

*For every Nash equilibrium there exists a corresponding Correlated Equilibrium.*
Computing Correlated Equilibrium

Computing a Correlated equilibrium is easier compared to Nash and can be found by linear programming even in general-sum case:

\[
\sum_{a_i \in A_i} -u_i(a_i, a_{-i}) \sigma(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \sigma(a_i, a_{-i}) \sum_{a_i \in A_i} \sigma(a_i) = 1
\]

\[
\sigma(a_i) \geq 0 \quad \forall a_i \in A_i
\]

\[
\forall i \in N, \forall a_i, a'_i \in A_i
\]

\[
\sum_{a \in A} \sigma(a) = 1
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\]

\[
\sum_{a \in A} \sigma(a) = 1 \quad \sigma(a) \geq 0 \quad \forall a \in A
\]
Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents. The leader—publicly commits to a strategy the follower(s) play a Nash equilibrium with respect to the commitment of the leader. Stackelberg equilibrium is a strategy profile that satisfies the above conditions and maximizes the expected utility value of the leader:

$$\text{arg max}_{s \in S; \forall i \in N \{1\}} s_i \in BR_i(s) u_1(s)$$
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\[
\arg \max_{s \in S; \forall i \in N \setminus \{1\} s_i \in BR_i(s_{-i})} u_1(s)
\]
Consider the following game:

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<td>(6, 1)</td>
</tr>
<tr>
<td>D</td>
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payoff to player 1

payoff to player 2
Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; 1 is the leader, 2 is the follower. Baseline polynomial algorithm requires solving $|A_2|$ linear programs:

$\max \sum_{a_1 \in A_1} s_1(a_1) u_1(a_1, a_2) \geq \sum_{a_1 \in A_1} s_1(a_1) u_2(a_1, a_2') \forall a_2' \in A_2$ for each $a_2 \in A_2$ assuming $a_2$ is the best response of the follower.
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$$\sum_{a_1 \in A_1} s_1(a_1) = 1$$

one for each $a_2 \in A_2$ assuming $a_2$ is the best response of the follower.