Normal-Form Games

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Noncooperative Game Theory

- Single round games
  - Normal-form games
  - Extensive-form games
  - MAIDS, Congestion games

- Multiple round games
  - Repeated games
  - Stochastic games
Types of games

- Two-player vs n-player
- Zero-sum games vs general-sum games
- Sequential vs one-shot
- Perfect-information vs imperfect-information
- Finite vs infinite
Types of games

- Two-player vs n-player
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Normal-form games

- Players set $P = \{1, ..., n\}$
- Actions set $A = A_1 \times ... \times A_i$
- Utility functions $u = \langle u_1, ... u_n \rangle$, where $u_i : A \to \mathbb{R}$
Normal-form games

- Represented as n-dimensional matrix
- Every entry is n-dimensional tuple of utilities for every player
A pure strategy $s_i$ in normal-form games represents the choice of specific action $a \in A_i$ for player $i$

A mixed strategy $m_i$ is a strategy distribution over pure strategies

Strategy profile $s/m$ is a set of pure/mixed strategies, one for every player

Overloading of utility function $u(s_i, s_{-i})$, $u(m_i, m_{-i})$, $u(m)$
Why do we need Game Theory?
Approaches for reasoning about games

- Studying game structure/properties
  - Social welfare optimality
  - Pareto optimality
- Stable strategies (solution concepts)
  - Maxmin
  - Minmax
  - Nash equilibrium
  - Stackelberg equilibrium
  - Correlated equilibrium
- Computation helpers
  - Dominance
Social welfare

- Defined as

\[ WF = \sum_{i \in P} u_i(m) \]  \hspace{1cm} (1)

- Not stable against deviations
- Cooperative players
Pareto optimality

- Reasoning about outcomes
- Outcome $o$ pareto dominates outcome $o'$ iff
  \[
  \forall i \in P : o_i \geq o'_i \text{ and } \exists i \in P : o_i > o'_i
  \]  
  (2)
- Outcome $o$ is pareto optimal if it is not pareto dominated by any other outcome $o'$
Dominance

- **Strict dominance**
  - Strategy $s_i$ strictly dominates $s'_i$ iff
    \[
    \forall s_{-_i} \in S_{-_i} : u(s_i, s_{-_i}) > u(s'_i, s_{-_i})
    \]  
    (3)

- **Weak dominance**
  - Strategy $s_i$ weakly dominates $s'_i$ iff
    \[
    \forall s_{-_i} \in S_{-_i} : u(s_i, s_{-_i}) \geq u(s'_i, s_{-_i}) \text{ and } \exists s_{-_i} \in S_{-_i} : u(s_i, s_{-_i}) > u(s'_i, s_{-_i})
    \]  
    (4) (5)

- **Very weak dominance**
  - Strategy $s_i$ very weakly dominates $s'_i$ iff
    \[
    \forall s_{-_i} \in S_{-_i} : u(s_i, s_{-_i}) \geq u(s'_i, s_{-_i})
    \]  
    (6)
Nash equilibrium

- A strategy \( m_i^* \) is the best response to strategies \( m_{-i} \), written as \( m_i^* \in BR(m_{-i}) \) iff
  \[
  \forall m_i \in M u_i(m_i^*, m_{-i}) \geq u_i(m_i, m_{-i})
  \]  
  (7)

- Nash equilibrium
  - Strategy profile \( m = \{m_1, ..., m_n\} \) is a Nash equilibrium iff
    \[
    \forall i \in P : m_i \in BR(m_{-i})
    \]  
    (8)

- Stable against deviations of players as every player plays his best response to the strategies of the rest
- Assumes self-interested rational players
- Every finite game has a non-empty set of Nash equilibria
- Examples
Properties of NE

- Values in NE might differ
- Strategies not interchangeable
- Mistake of the opponent might hurt me
Properties of NE in zero-sum games

- All NE have the same value for $i$ (value of the game)
- The value is guaranteed (mistakes of the opponent only increase my expected outcome)
- Strategies are interchangeable between NE
- $\text{minmax} = \text{maxmin} = \text{NE} = \text{SE}$
LP for solving zero-sum NFG

\[
\begin{align*}
\max_{U_i, m_i(a)} U_i \\
\text{s.t. } \sum_{s_i \in S_i} u_i(s_i, s_{-i}) m_i(s_i) &\geq U_i, \quad \forall s_{-i} \in S_{-i} \\
\sum_{s_i \in S_i} m_i(s_i) &= 1 \\
m_i(s_i) &\geq 0, \quad \forall s_i \in S_i
\end{align*}
\]

All NE are feasible solutions of this LP