Resolution with Equality and Subsumption for first-order logic

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Resolution with Equality

- Equality is a common part of many problems for ATP.
- How to formalize equality in first-order logic?

- Gottfried Wilhelm Leibnitz (1646-1716):

  Given any $X$ and $Y$, $X = Y$
  if and only if
  given any predicate $p$, $p(X) \iff p(Y)$. 
Resolution with Equality

A *standard method* for explicitly adding equality is to add properties of equality as axioms to an input problem:

1. **Reflexivity:** \((\forall X) (X=X)\)
2. **Symmetry:** \((\forall X) (\forall Y) (X=Y \rightarrow Y=X)\)
3. **Transitivity:** \((\forall X) (\forall Y) (\forall Z) ((X=Y \land Y=Z) \rightarrow X=Z)\)
4. **Congruence:**

   for every function symbol in the problem introduce:
   \[(\forall X_1...X_n) (\forall Y_1...Y_n) ((X_1=Y_1 \land ... \land X_n=Y_n) \rightarrow (f(X_1,...,X_n) = f(Y_1,...,Y_n)))\]

   for every predicate symbol in the problem introduce:
   \[(\forall X_1...X_m) (\forall Y_1...Y_m) ((X_1=Y_1 \land ... \land X_m=Y_m) \rightarrow (p(X_1,...,X_m) \rightarrow p(Y_1,...,Y_m)))\]
Problems of The Standard Method

- More predicate and functional symbols we have, more axioms/clauses we need to add.
- The ATP complexity is progressing very fast.
- Even a very simple problem can have very complicated proof.

Example:

$$(X=Y) \rightarrow (p(f(g(X))) \rightarrow p(f(g(Y))))$$
Paramodulation

- **paramodulation rule**: 

\[
\frac{A \lor (s = t) \quad B[u]}{A\theta \lor B\theta[t\theta]}
\]

where:

- $\Theta$ is MGU of terms $s$ and $u$
- $(s = t)$ denotes $t = s$ or $s = t$
- $B\theta[t\theta]$ is obtained by replacing a single occurrence of $u\theta$ in $B\theta$ by $t\theta$
- the inferred clause is called *binary paramodulant*
Paramodulation

- reflexivity resolution rule:

\[
A \lor \neg(s = t) \\
\hline
A\Theta
\]

where:

- \( \Theta \) is MGU of terms \( s \) and \( t \)
- \( (s = t) \) denotes \( t = s \) or \( s = t \)
Paramodulation

Theorem: Paramodulation composed by
- resolution rule
- factorization rule
- paramodulation rule
- reflexivity resolution rule

forms a formal system that is refutation-complete

(it is able to derive false from every unsatisfiable set of formulae with equality)
Subsumption

- A clause $C$ *subsumes* a clause $D$ iff there is a substitution $\theta$ such that $C\theta \subseteq D$ (where clause is viewed as the set of its literals). $D$ is called a *subsumed* clause.

- We denote $C \sqsubseteq D$

- If $C \sqsubseteq D$ then $C \vDash D$.

- For set of clauses $A$ and $B$ we call that $A$ subsumes $B$ ($A \sqsubseteq B$) iff for each clause $B \in B$ exists $A \in A$ such that $A \sqsubseteq B$.

- Examples:
  
  $p(X) \sqsubseteq p(c)$  
  $p(X) \sqsubseteq p(X) \lor q(X,Y)$  
  $p(X) \sqsubseteq p(f(c)) \lor r(X,f(Y))$  
  $p(X) \lor q(Z) \sqsubseteq p(X) \lor q(X)$  
  $p(X) \lor q(X) \not\sqsubseteq p(X) \lor q(Z)$
Subsumption in Resolution

- If we have a resolution proof (by refutation) of set $\mathcal{B}$ and if holds $\mathcal{A} \subseteq \mathcal{B}$ then there exists a resolution proof of set $\mathcal{A}$ that is not longer than the proof of $\mathcal{B}$.

- Idea of proof:
  - All clauses in $\mathcal{A}$ have at least the same “power” as clauses in $\mathcal{B}$.
  - Hence, the proof of $\mathcal{A}$ cannot be “harder” than the proof of $\mathcal{B}$.

- Usage:
  - If we can infer two clauses $\mathcal{A}$ and $\mathcal{B}$ where it holds $\mathcal{A} \subseteq \mathcal{B}$ then we can keep $\mathcal{A}$ only and $\mathcal{B}$ can be removed without loss of refutation completeness.
Consider the following resolution proof of 
\{p(f(X)) \lor q(X,Y) \lor r(X), \neg p(f(f(c))), \neg q(U,V), \neg r(f(c))\}:

\[
\begin{array}{c}
p(f(X)) \lor q(X,Y) \lor r(X) \quad \neg p(f(f(c))) \\
\hline
q(f(c),Y) \lor r(f(c)) \\
\hline
r(f(c))
\end{array}
\]

\[
\neg q(U,V) \quad \neg r(f(c))
\]

If we can infer a clause \(p(Y) \lor r(Z)\) somewhere in the proof then we can replace \(p(f(X)) \lor q(X,Y) \lor r(X)\) by \(p(Y) \lor r(Z)\) (Because \(p(Y) \lor r(Z) \equiv p(f(X)) \lor q(X,Y) \lor r(X)\))

Now we can update/simplify the previous proof:

\[
\begin{array}{c}
p(Y) \lor r(Z) \quad \neg p(f(f(c))) \\
\hline
r(Z) \\
\hline
\neg r(f(c))
\end{array}
\]