Statistical Machine Learning (BE4M33SSU)
Lecture 9: Hidden Markov Models

Czech Technical University in Prague

- Markov Models and Hidden Markov Models
- Inference algorithms for HMMs
- Parameter learning for HMMs

1. Structured hidden states

Models discussed so far: mainly classifiers predicting a categorical (class) variable $y \in \mathcal{Y}$

Often in applications: the hidden state is a structured variable.

Here: the hidden state is given by a **sequence** of categorical variables.

Application examples:

- text recognition (printed, handwritten, "in the wild"),
- speech recognition (single word recognition, continuous speech recognition, translation),
- robot self localisation.

Markov Models and Hidden Markov Models on chains:
a class of generative probabilistic models for sequences of features and sequences of categorical variables.

2. Markov Models

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ denote a sequence of length n with elements from a finite set K.

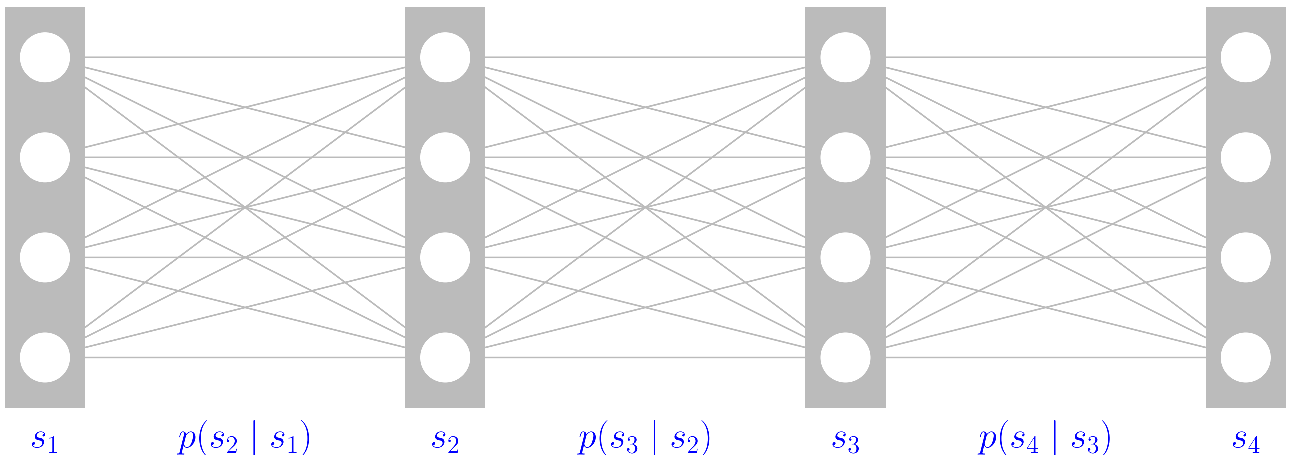
Any joint probability distribution on K^n can be written as

$$p(s_1, s_2, \ldots, s_n) = p(s_1)\, p(s_2 \mid s_1)\, p(s_3 \mid s_2, s_1) \cdot \ldots \cdot p(s_n \mid s_1, \ldots, s_{n-1})$$

Definition 1 A joint p.d. on K^n is a Markov model if

$$p(\boldsymbol{s}) = p(s_1)\, p(s_2 \mid s_1)\, p(s_3 \mid s_2) \cdot \ldots \cdot p(s_n \mid s_{n-1}) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1})$$

holds for any $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$.



2. Markov Models

Example 1 (Random walk on a graph)

- Let (V, E) be a directed graph. A random walk in (V, E) is described by a sequence $\boldsymbol{s} = (s_1, \ldots, s_t, \ldots)$ of visited nodes, i.e. $s_t \in V$.
- The walker starts in node $i \in V$ with probability $p(s_1 = i)$.
- The edges of the graph are weighted by $w \colon E \to \mathbb{R}_+$, s.t.

$$\sum_{j \colon (i,j) \in E} w_{ij} = 1 \quad \forall i \in V$$

- In the current position $s_t = i$, the walker randomly chooses an outgoing edge with probability given by the weights and moves along this edge, i.e.

$$p(s_{t+1} = j \mid s_t = i) = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Questions: How does the distribution $p(s_t)$ behave? Does it converge to some fix-point distribution for $t \to \infty$?

3. Algorithms: Computing the most probable sequence

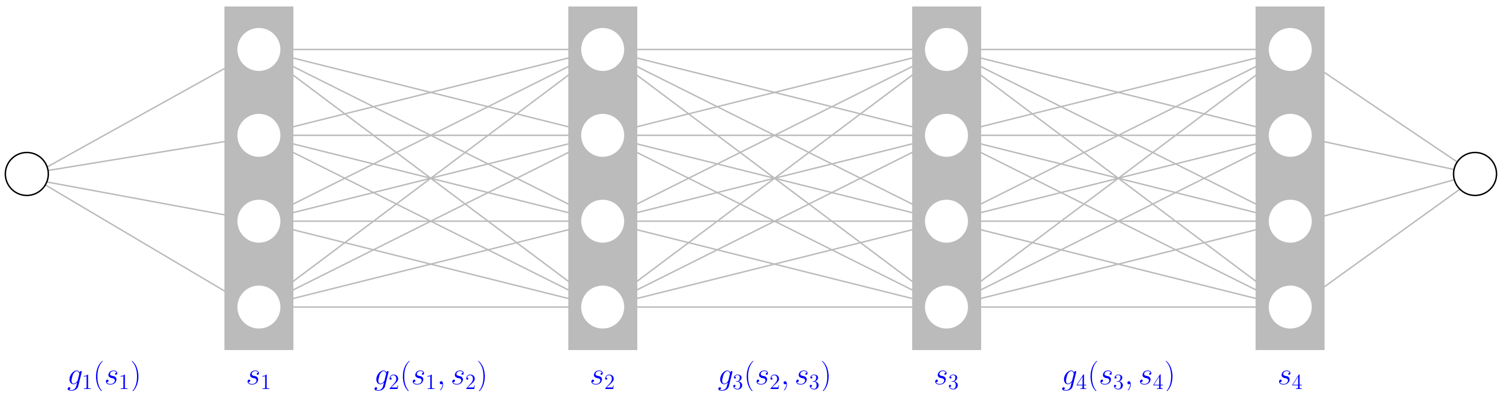
How to compute the most probable sequence $\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1})$?

Take the logarithm of $p(\boldsymbol{s})$: $\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} \big[g_1(s_1) + \sum_{i=2}^{n} g_i(s_{i-1}, s_i)\big]$

and apply dynamic programming: Set $\phi_1(s_1) \equiv g_1(s_1)$ and compute

$$\phi_i(s_i) = \max_{s_{i-1}\in K} \big[\phi_{i-1}(s_{i-1}) + g_i(s_{i-1}, s_i)\big].$$

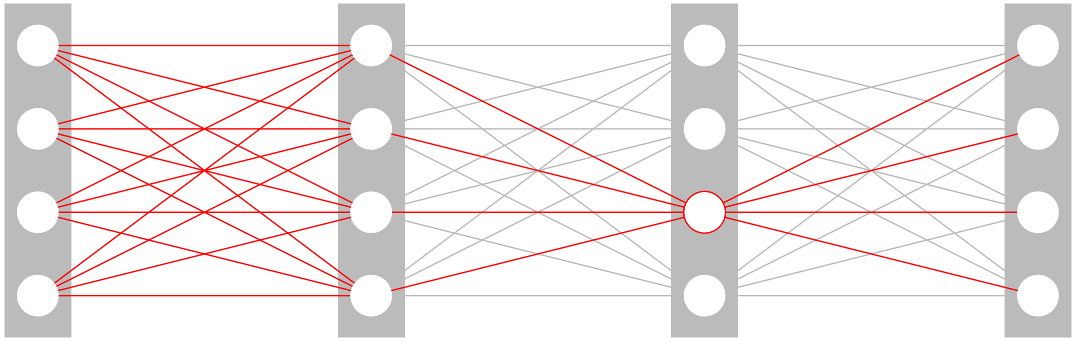
Finally, find $s_n^* \in \arg\max_{s_n\in K} \phi_n(s_n)$ and back-track the solution. This corresponds to searching the best path in the graph



3. Algorithms: Computing marginal probabilities

How to compute marginal probabilities for the sequence element s_i in position i

$$p(s_i) = \sum_{s_1 \in K} \cdots \sum_{s_i \in K} \cdots \sum_{s_n \in K} p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1})$$



Summation over the trailing variables is easily done because:

$$\sum_{s_n \in K} p(s_1) \cdots p(s_{n-1} \mid s_{n-2})\, p(s_n \mid s_{n-1}) = p(s_1) \cdots p(s_{n-1} \mid s_{n-2})$$

The summation over the leading variables is done dynamically: Begin with $p(s_1)$ and compute

$$p(s_i) = \sum_{s_{i-1} \in K} p(s_i \mid s_{i-1})\, p(s_{i-1})$$

3. Algorithms: Computing marginal probabilities

This computation is equivalent to a matrix vector multiplication: Consider the values $p(s_i = k \mid s_{i-1} = k')$ as elements of a matrix $P_{k'k}(i)$ and the values of $p(s_i = k)$ as elements of a vector $\boldsymbol{\pi}_i$. Then the computation above reads as $\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i-1} P(i)$.

Remark 1

- Notice that the preferred direction (from first to last) in the Definition 1 of a Markov model is only apparent. By computing the marginal probabilities $p(s_i)$ and by using $p(s_i \mid s_{i-1}) p(s_{i-1}) = p(s_{i-1}, s_i) = p(s_{i-1} \mid s_i) p(s_i)$, we can rewrite the model in reverse order.
- A Markov model is called homogeneous if the transition probabilities $p(s_i = k \mid s_{i-1} = k')$ do not depend on the position i in the sequence. In this case the formula $\boldsymbol{\pi}_i = \boldsymbol{\pi}_1 P^{i-1}$ holds for the computation of the marginal probabilities.

3. Algorithms: Learning a Markov model

Suppose we are given i.i.d. training data $\mathcal{T}_m = \{\boldsymbol{s}^j \in K^n \mid j = 1, \ldots, m\}$ and want to estimate the parameters of the Markov model by the maximum likelihood estimate. This is very easy:

- Denote by $\alpha(s_{i-1} = \ell, s_i = k)$ the fraction of sequences in $\mathcal{T}_m$ for which $s_{i-1} = \ell$ and $s_i = k$.
- The estimates for the conditional probabilities are then given by

$$p(s_i = k \mid s_{i-1} = \ell) = \frac{\alpha(s_{i-1} = \ell, s_i = k)}{\sum_k \alpha(s_{i-1} = \ell, s_i = k)}.$$

4. Hidden Markov Models

- Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ denote a sequence of hidden states from a finite set K.
- Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denote a sequence of features from some feature space $\mathcal{X}$.

Definition 2 A joint p.d. on $\mathcal{X}^n \times K^n$ is a Hidden Markov model if

(a) the prior p.d. $p(\boldsymbol{s})$ for the sequences of hidden states is a Markov model, and

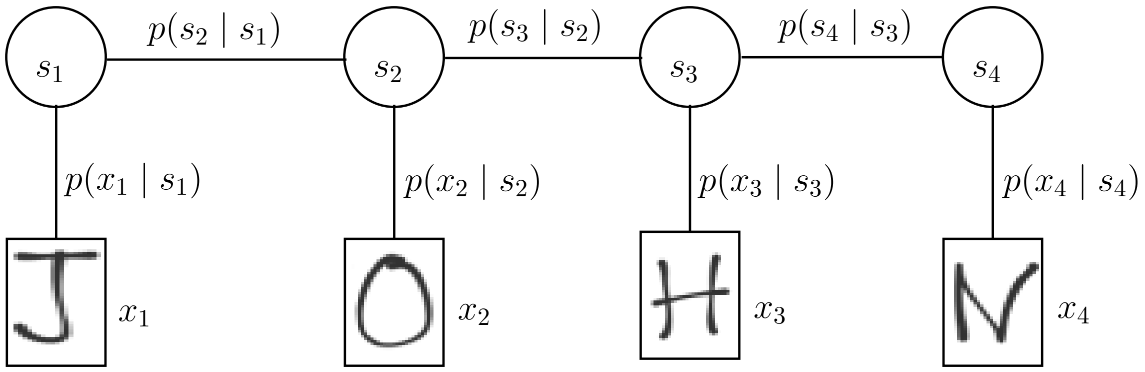
(b) the conditional distribution $p(\boldsymbol{x} \mid \boldsymbol{s})$ for the feature sequence is independent, i.e.

$$p(\boldsymbol{x} \mid \boldsymbol{s}) = \prod_{i=1}^{n} p(x_i \mid s_i).$$

Example 2 (Text recognition, OCR)

- $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ – sequence of images with characters,
- $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ – sequence of alphabetic characters,
- $p(s_i \mid s_{i-1})$ – language model,
- $p(x_i \mid s_i)$ – appearance model for characters.

Hidden Markov Models



4. Algorithms for HMMs

(1) Find the most probable sequence of hidden states given the sequence of features:

$$\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} \; p(s_1)\prod_{i=2}^{n} p(s_i \mid s_{i-1}) \prod_{i=1}^{n} p(x_i \mid s_i)$$

Take the logarithm, redefine the g-s and apply dynamic programming as before for Markov models.

(2) Compute marginal probabilities for hidden states given the sequence of features:

This is now more complicated, because we need to sum over the leading and trailing hidden state variables. Do this by dynamic matrix-vector multiplication from the left and from the right

$$\phi_i(s_i) = \sum_{s_{i-1}} p(x_i \mid s_i)\, p(s_i \mid s_{i-1})\, \phi_{i-1}(s_{i-1})$$

$$\psi_i(s_i) = \sum_{s_{i+1}} p(x_{i+1} \mid s_{i+1})\, p(s_{i+1} \mid s_i)\, \psi_{i+1}(s_{i+1})$$

4. Algorithms for HMMs

The (posterior) marginal probabilities are then obtained from

$$p(s_i \mid \boldsymbol{x}) \sim \phi_i(s_i)\psi_i(s_i)$$

The computational complexity is $\mathcal{O}(nK^2)$.

(3) Learning the model parameters from training data:

Given i.i.d. training data $\mathcal{T}_m = \{(\boldsymbol{x}^j, \boldsymbol{s}^j) \in \mathcal{X}^n \times K^n \mid j = 1, \ldots, m\}$, estimate the parameters of the HMM by the maximum likelihood estimator.

This is done by simple "counting" as before for Markov models.