Dynamic programming

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We want to find a maximal cost sequence

$$(y_1^*, \dots, y_L^*) \in \operatorname*{Argmax}_{(y_1,\dots,y_L)\in\mathcal{Y}^L} \left[\sum_{i=1}^{L} q_i(y_i) + \sum_{i=1}^{L-1} g(y_i, y_{i+1}) \right] \qquad (1)$$

where $L \in \mathbb{N}$ is a length of the sequence, $\mathcal{Y}$ is a finite set of labels, $q_i \colon \mathcal{Y} \to \mathbb{R}$, $i \in \{1, \dots, L\}$, and $g \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ are some fixed functions.

Let $F_i \colon \mathcal{Y} \to \mathbb{R}$, $i \in \{1, \dots, L\}$, and $L_i \colon \mathcal{Y} \to \mathcal{Y}$, $i \in \{2, \dots, L\}$, be functions defined recursively as follows:

$$\begin{aligned}
F_1(y) &= q_1(y)\,, \qquad \forall y \in \mathcal{Y}\,, \\
F_i(y) &= q_i(y) + \max_{y'\in\mathcal{Y}} \Big[F_{i-1}(y') + g(y', y) \Big]\,, \qquad i = 2, \dots, L \\
L_i(y) &\in \operatorname*{Argmax}_{y'\in\mathcal{Y}} \Big[F_{i-1}(y') + g(y', y) \Big]\,, \qquad i = 2, \dots, L
\end{aligned}$$

Then, the optimal value of the problem (1) equals to

$$F^* = \max_{y\in\mathcal{Y}} F_L(y)\,,$$

and the last label in the optimal sequence is

$$y_L^* \in \operatorname*{Argmax}_{y\in\mathcal{Y}} F_L(y)\,.$$

The other labels in the optimal sequence are found recursively as follows

$$y_{i-1}^* = L_i(y_i^*)\,, \qquad i = L-1, \dots, 2\,.$$