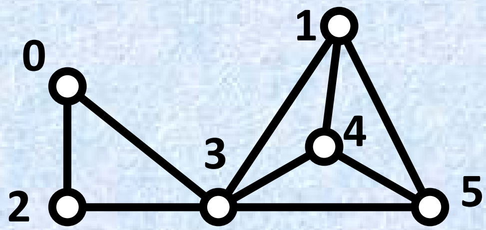
Graph



- **Nodes, Vertices**
- Servers, cities...
- Persons, people...
- Objects in comp. science
- ... etc.

- **Edges**
- Connections, roads...
- Personal relations
- Relations among objects
- ... etc.

Usual graph representations

nodes = indices		**Node degrees**		**Lists of neighbours**				
0	.....	2	.....	2	3			
1	.....	3	.....	3	4	5		
2	.....	2	.....	0	3			
3	.....	5	.....	0	2	1	4	5
4	.....	3	.....	1	3	5		
5	.....	3	.....	1	3	4		

1D/2D array, vector, ArrayList...

Adjacency matrix

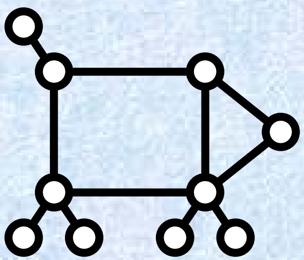
Nodes = indices	0	1	2	3	4	5
0	0	0	1	1	0	0
1	0	0	0	1	1	1
2	1	0	0	1	0	0
3	1	1	1	0	1	1
4	0	1	0	1	0	1
5	0	1	0	1	1	0

2D array, matrix

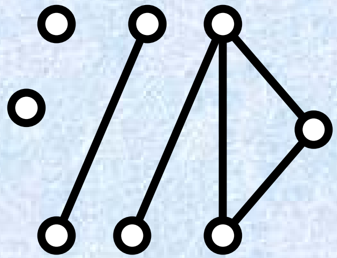
Less obvious, more effective

Plain, obvious, less effective

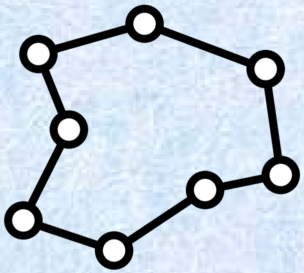
Small graph zoo



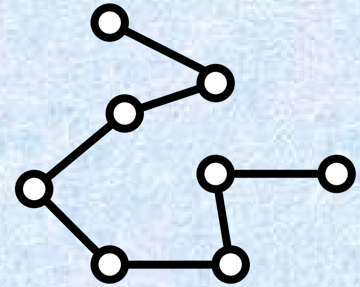
- **Connected graph**



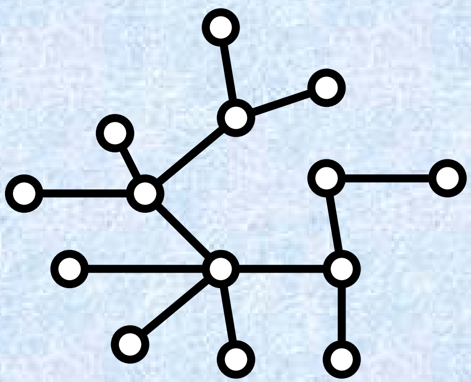
- **Disconnected**
- **graph**



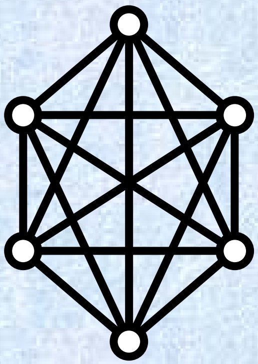
- **Cycle / circle**
- N nodes,
  N edges



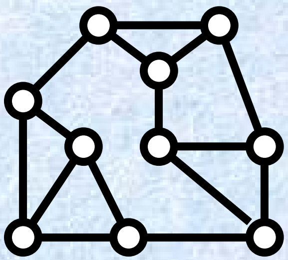
- **Path**
- N nodes,
  N–1 edges



- **Tree**
- Connected
- N nodes, N–1 edges
- is bipartite

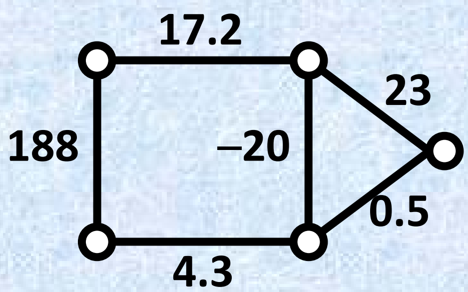


- **Complete graph**
- N nodes
- $(N^2 – N)/2$ edges

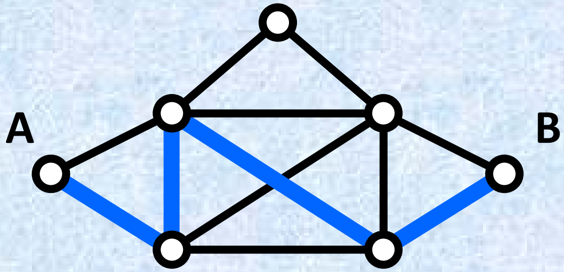


- **Regular graph**
- All node degrees
  are the same

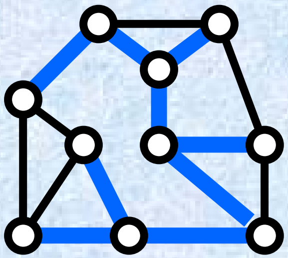
Small graph zoo



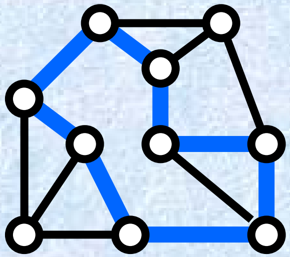
- **Weighted graph**
- Each edge has its cost (length, weight)



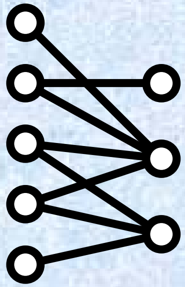
- **Path between A and B**
- Path visits each node at most once



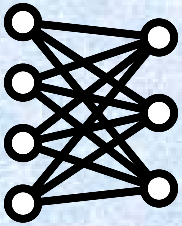
- **Spanning tree**
- subgraph which is a tree and it contains all nodes



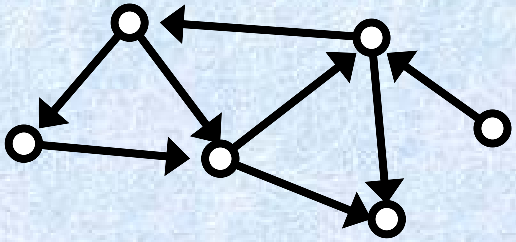
- **Cycle in a graph**
- path which first and last node are the same



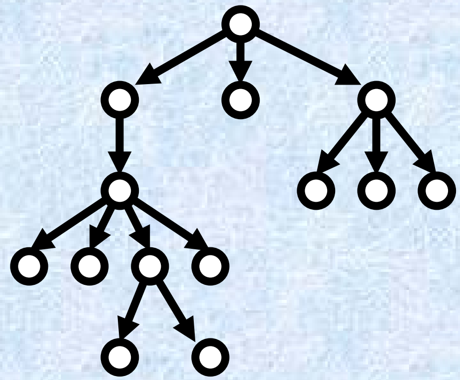
- **Bipartite graph**
- two-colorable
- cycles only of even length
- No edges inside partitions



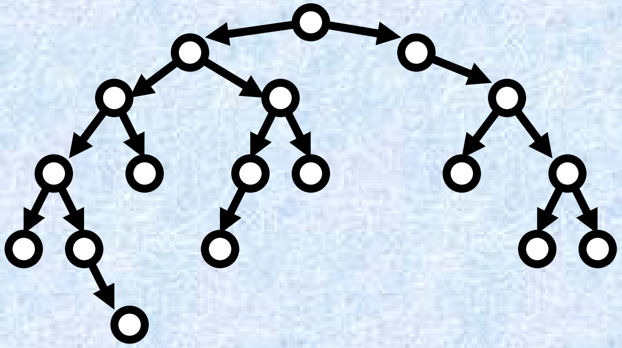
- **Complete bipartite graph**
- M and N nodes in partitions
- M x N edges



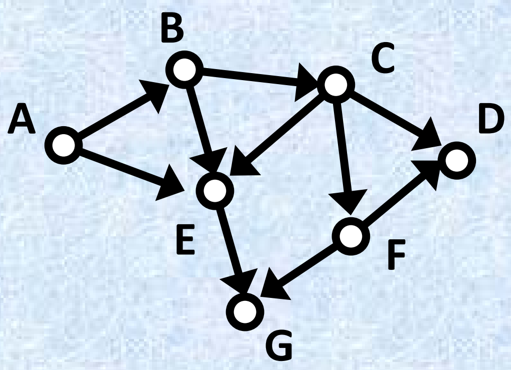
❖ **Directed graph**



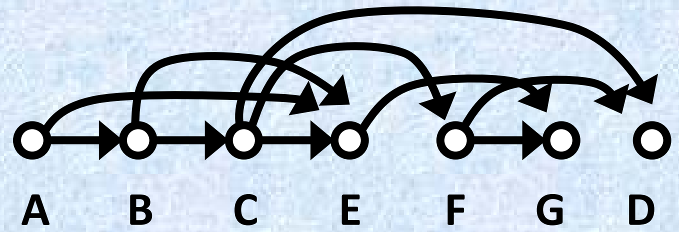
❖ **Rooted tree**



❖ **Binary rooted tree**



❖ **Directed acyclic graph (DAG)**
❖ No directed loops



❖ **Topological order** of the same DAG

A few apparently innocuous problems related to graphs

A few apparently innocuous problems related to graphs

Easy problem = a complete solution may be taught in bachelor courses.

Hard problem = a complete solution is unknow to this day.
(However, there often exist satisfactory approximate solutions.
Typically, they are quite advanced)

Clay Mathematics Institute
http://www.claymath.org/millennium-problems/rules-millennium-prizes
Offers prize **1 000 000 $** for a complete solution of any of those hard questions.
The prize exists since the year 2000.
Nobody has claimed it yet :-(...

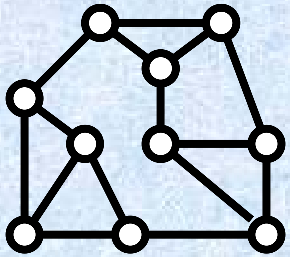
Connectivity

Is there a path between any two nodes?

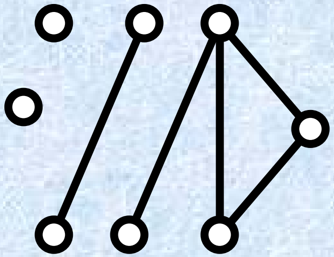
Easy problem

Algorithm: DFS, BFS, Union-Find

Complexity: DFS, BFS $O(|V|+|E|)$, Union-Find $O(|E| \cdot \alpha(|V|))$



Yes,
one connected component.



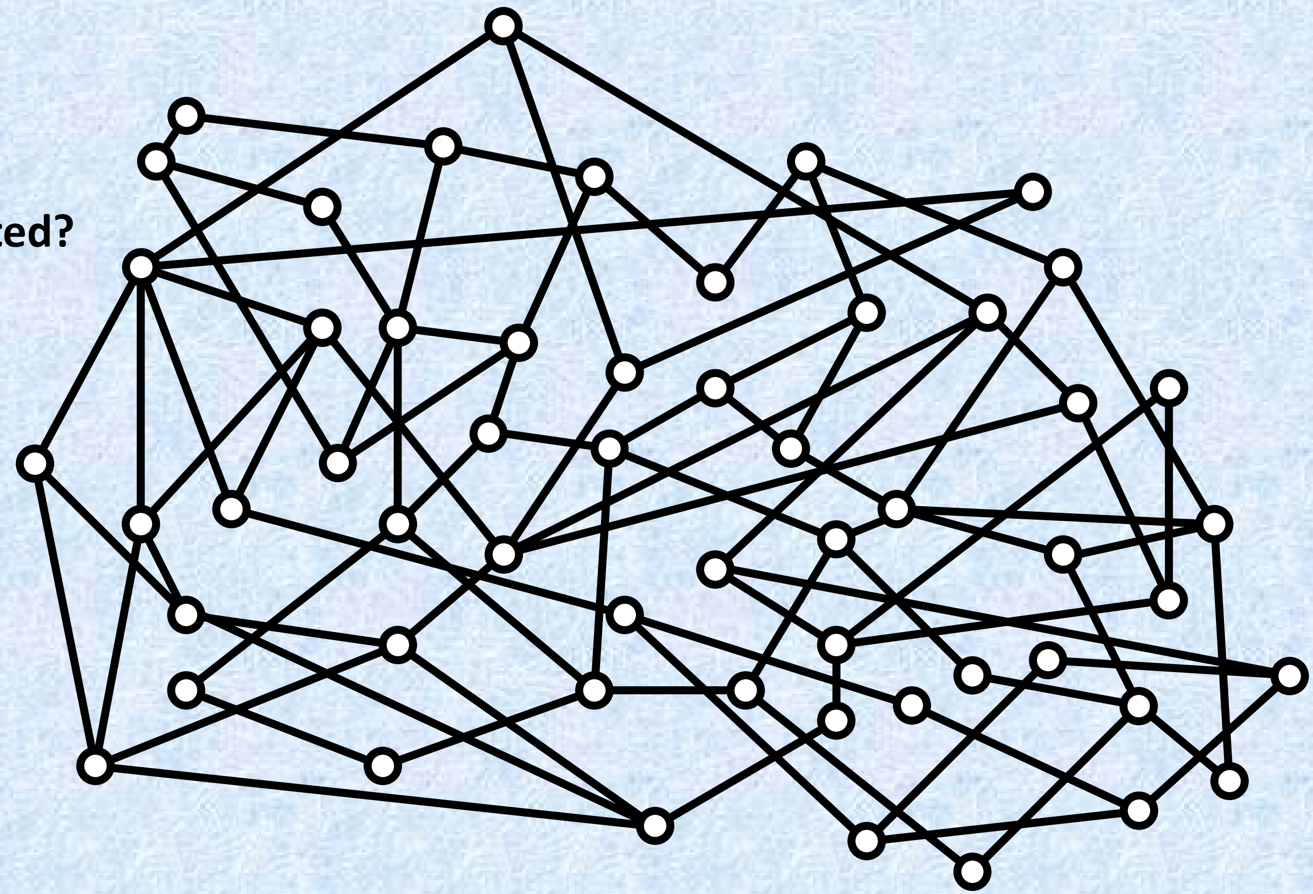
No,
four connected components.

Connectivity

Is there a path between any two nodes?

Easy problem

Is the graph connected?



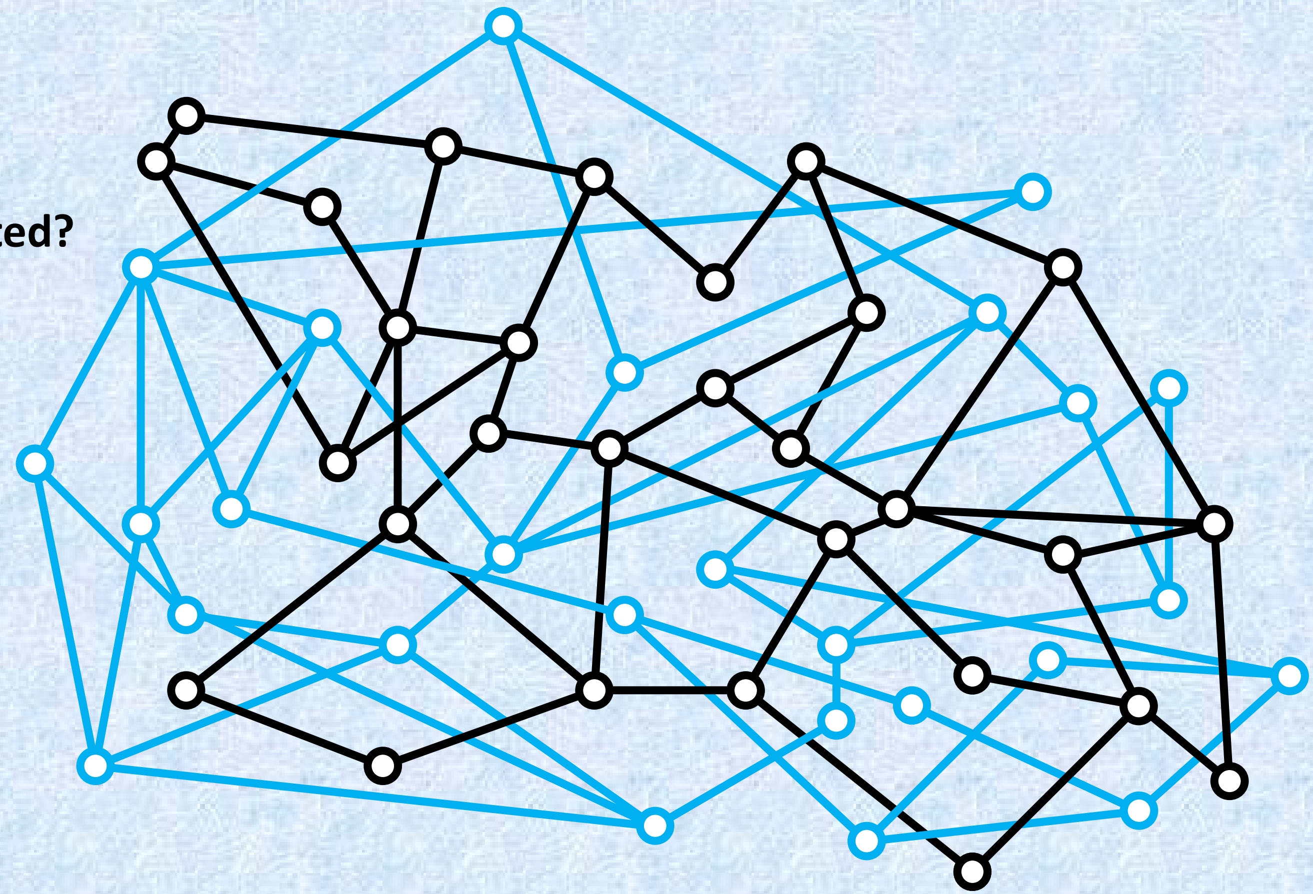
Connectivity

Is there a path between any two nodes?

Easy problem

Is the graph connected?

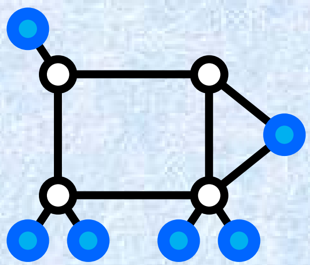
No,
it consists of
two components.

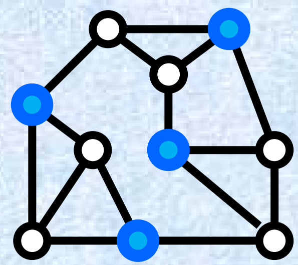


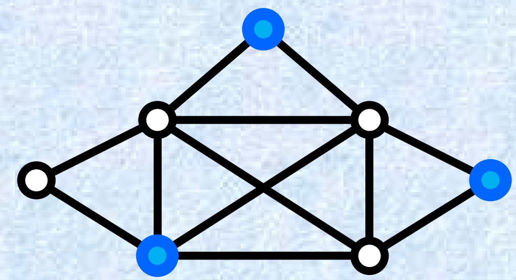
Independence

Maximum size of a set of nodes in which no two nodes are adjacent.

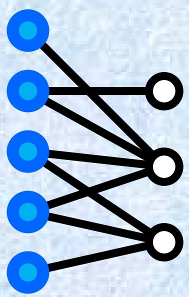
Hard problem in general

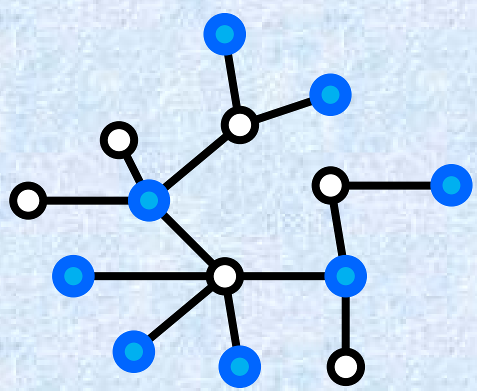


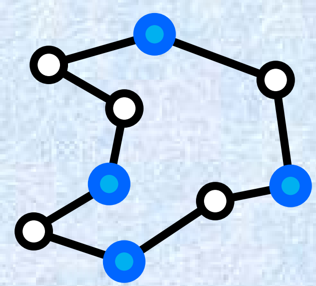


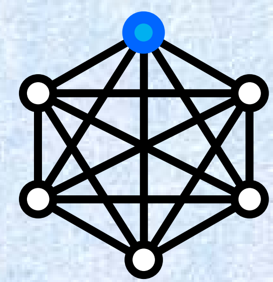


Easy problem on graphs with some particular structure









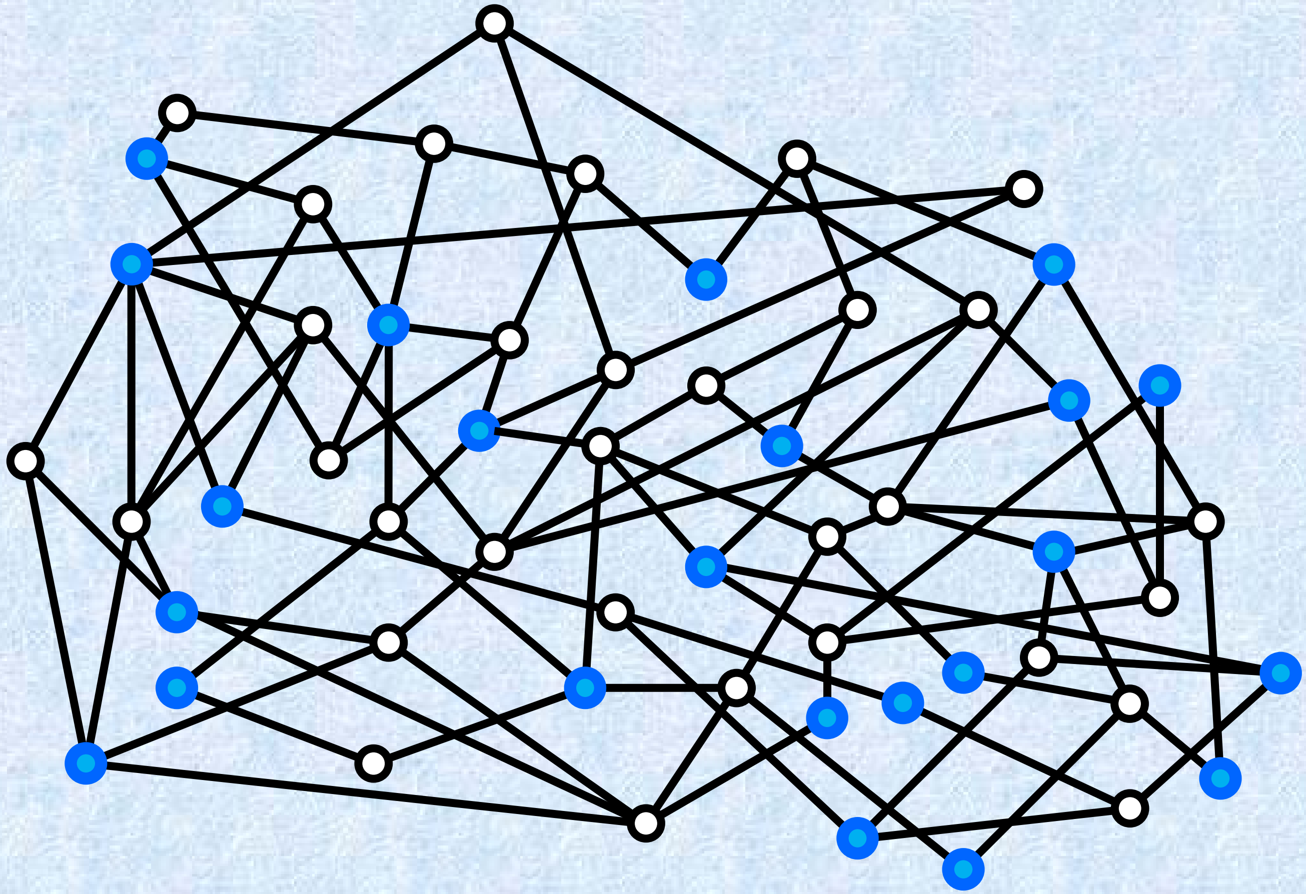
- **Bipartite graph**
- **Tree** is always bipartite
- **Cycle**
- **Complete graph**

Independence

Maximum size of a set of nodes in which no two nodes are adjacent.

Ex: How many of them in this graph? more than 23?

Hard problem

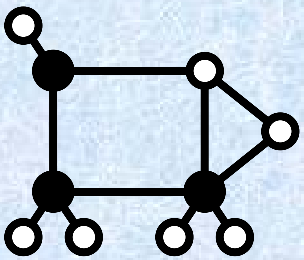


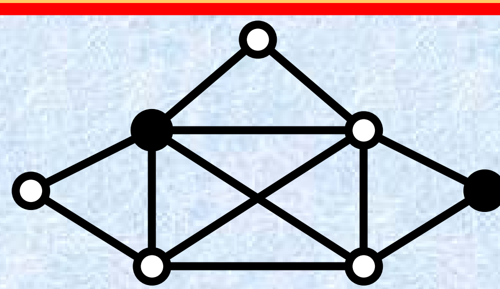
Dominance

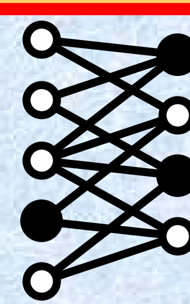
Maximum size of such set M of nodes that each node in the graph is either in M or is a neighbour of some node in M.

Ex. A fire station must be located either in a village or in the immediately neighbour village. How many fire stations are enough to serve the region?

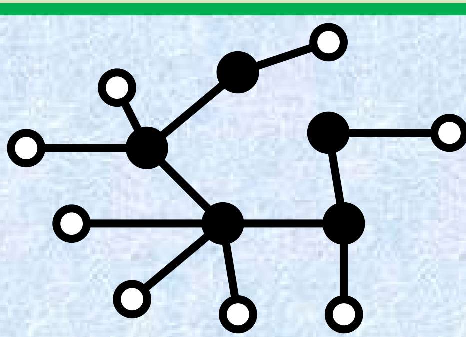
Hard problem

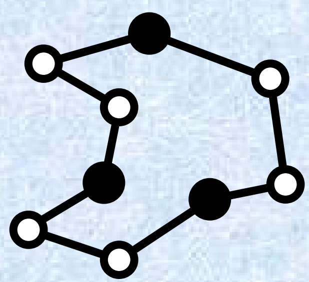


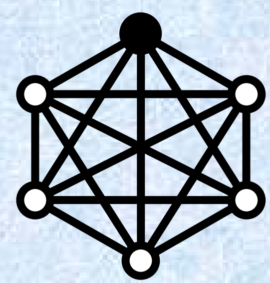




Easy problem on graphs with some particular structure





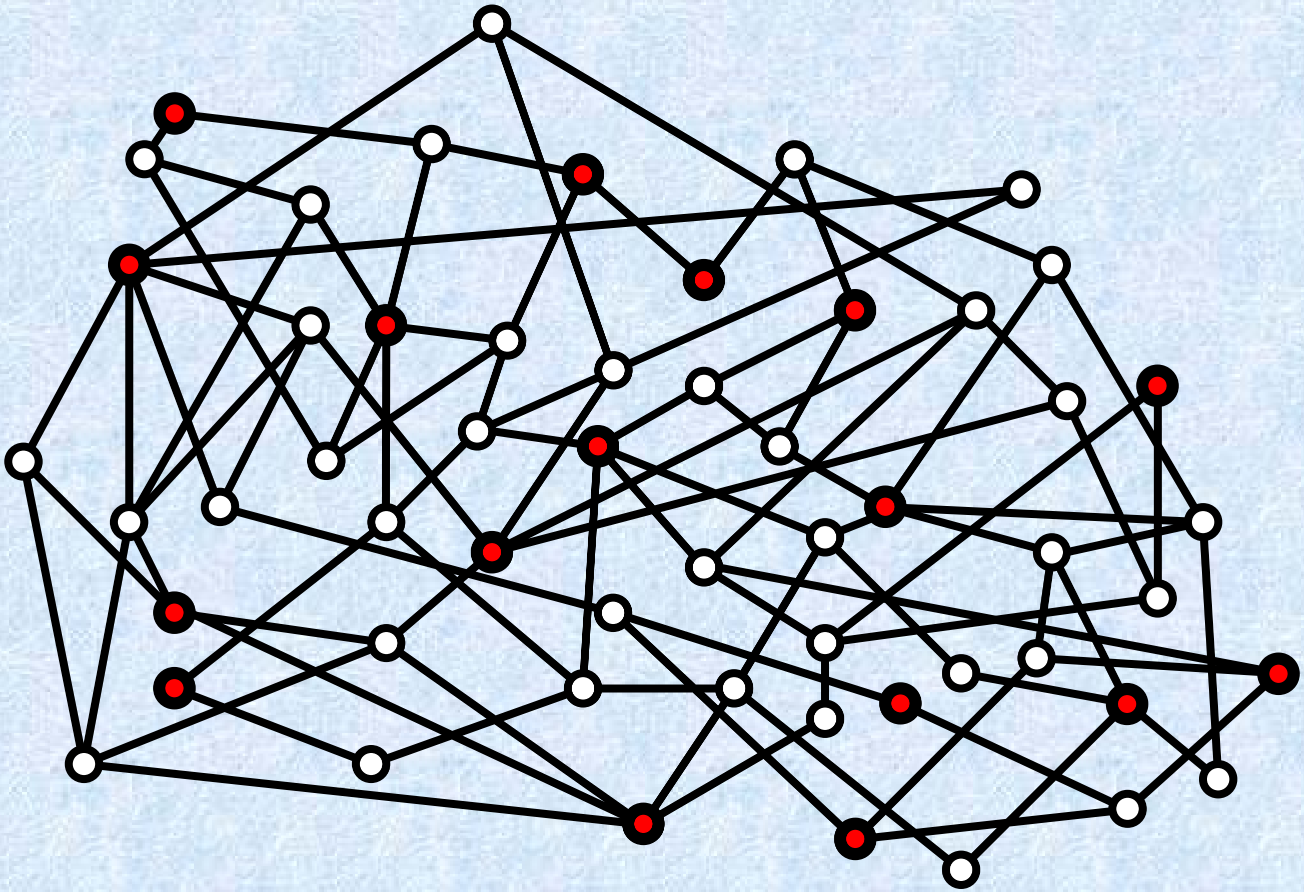


- **Tree,** apply Dynamic programming
- **Circle**
- **Complete graph**

Dominance

Ex. A fire station must be located either in a village or in the immediately neighbour village. Can there be less than 17 fire stations to serve the region?

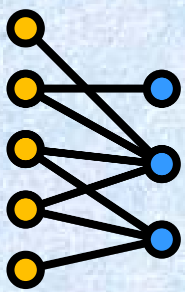
Hard problem



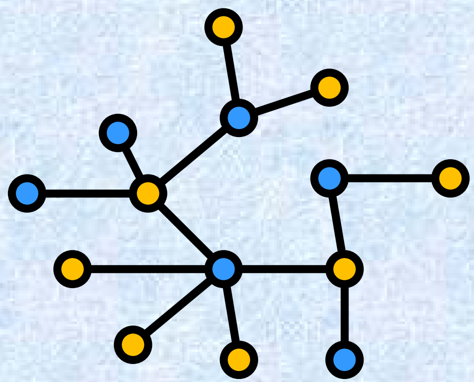
Colorability, chromatic number

Minimum number of colors needed to color each node so that any two neighbours have different color.

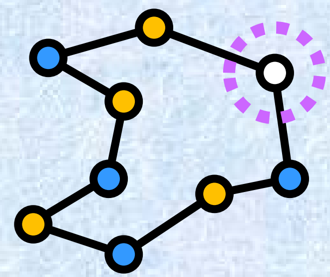
Is 2 colors enough? -- Easy problem.   Graph must be bipartite.



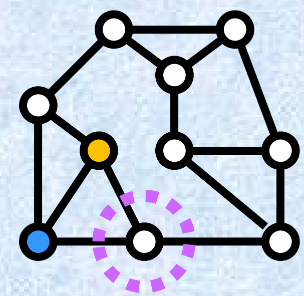
2 colors , bipartite graph



2 colors for any tree



2 colors are not enough in a cycle of odd length.



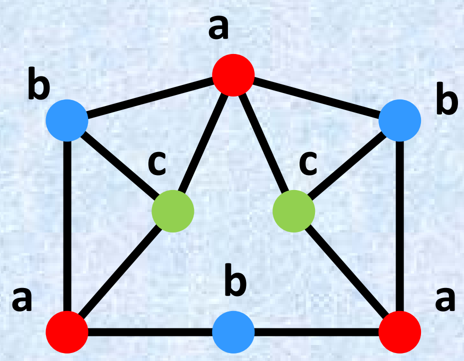
2colors are not enough, there is a cycle of odd length in the graph

Is graph bipartite? Apply BFS.
Mark by 1 all nodes in odd distance from the start and mark by 0 all nodes in even distance from start. If any two nodes with the same mark are connected by an edge, the graph is not bipartite (two-colorable).

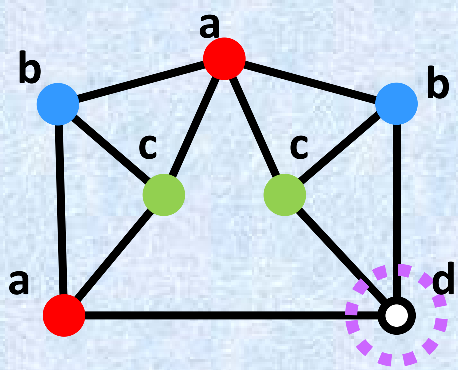
Colorability, chromatic number

Minimum number of colors needed to color each node so that any two neighbours have different color.

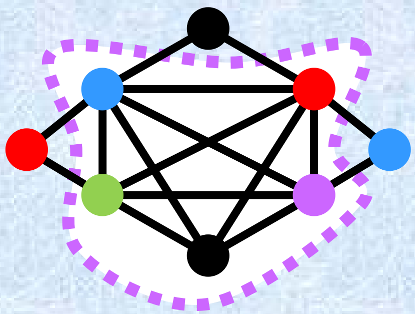
Hard problem -- Are 3 colors enough?



3 colors suffice



4 colors.
The node colors are chosen WLOG, the color of node at the bottom right cannot be any of a, b, c.



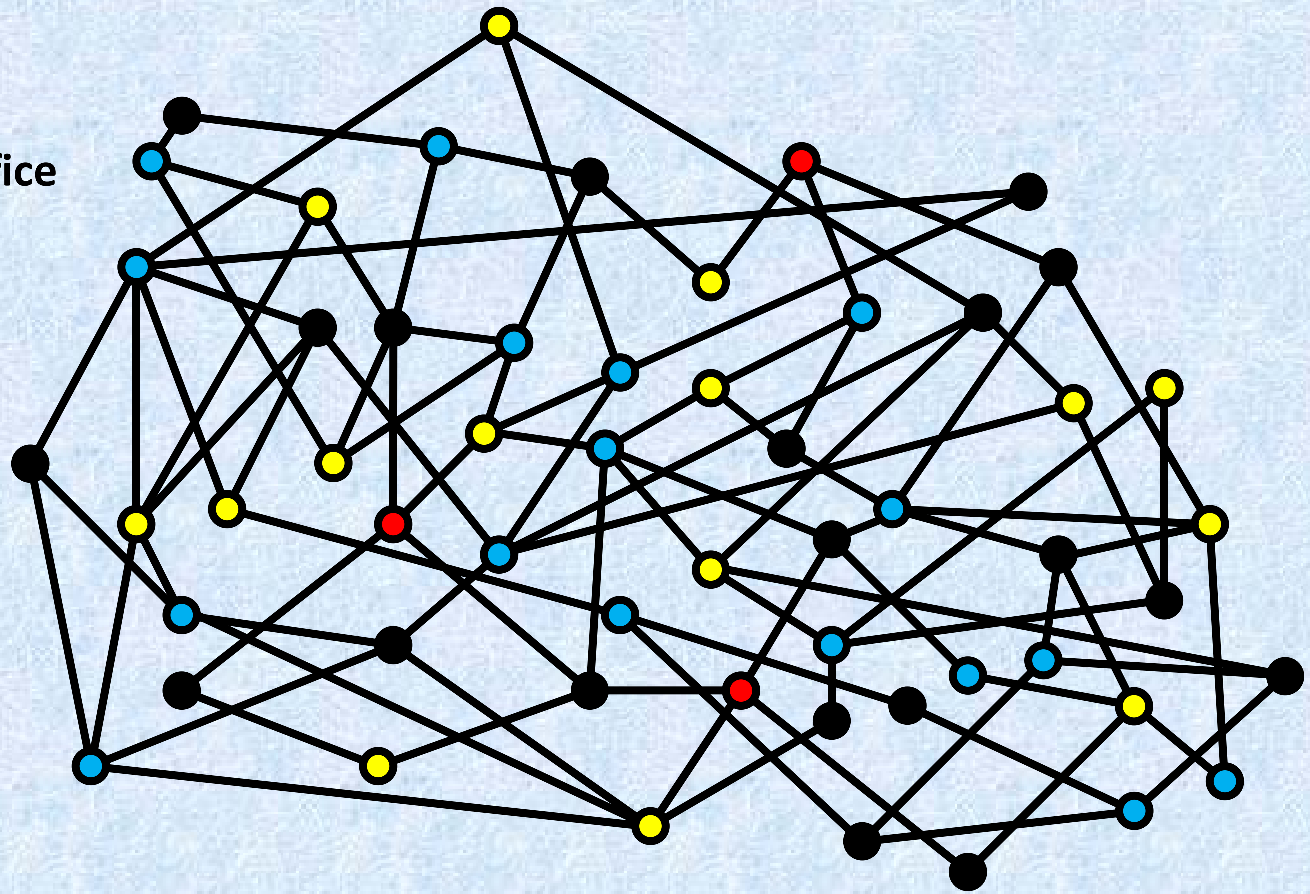
5 colors.
The graph contains a clique (complete subgraph) of size 5.
Clique detection is a hard problem.

Colorability, chromatic number

Minimum number of colors needed to color each node so that any two neighbours have different color.

Hard problem -- Are 3 colors enough?

4 colors are suffice in this graph. Maybe 3 colors would suffice too? ... ??



Shortest paths

Minimum possible number of edges (nodes) on a path from A to B.

Easy problem

Algorithms: BFS, Dijkstra, Bellman–Ford, Floyd–Warshall, Johnson...
Complexities: Polynomial, mostly less than O($|V|^3$).

Longest paths

Typically, each node/edge can be visited at most once.

Hard problem for general graphs

Easy problem for trees and DAGs

Algorithm: Dynamic programming
Compexity: O($|V|+|E|$)

Minimum spanning tree

Minimum total cost (weight) of selected edges which connect all nodes in the graph. The selected edges form a tree.

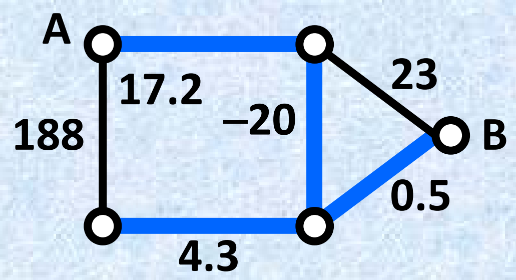
Easy problem

Algorithms: Prim's $O(|V|^2)$ with matrix representation

$O(|E| \cdot \log(|V|))$ with linked list representation and with binary heap

Kruskal's $O(|E| \cdot \log(|V|))$

Borůvka's $O(|E| \cdot \log(|V|))$

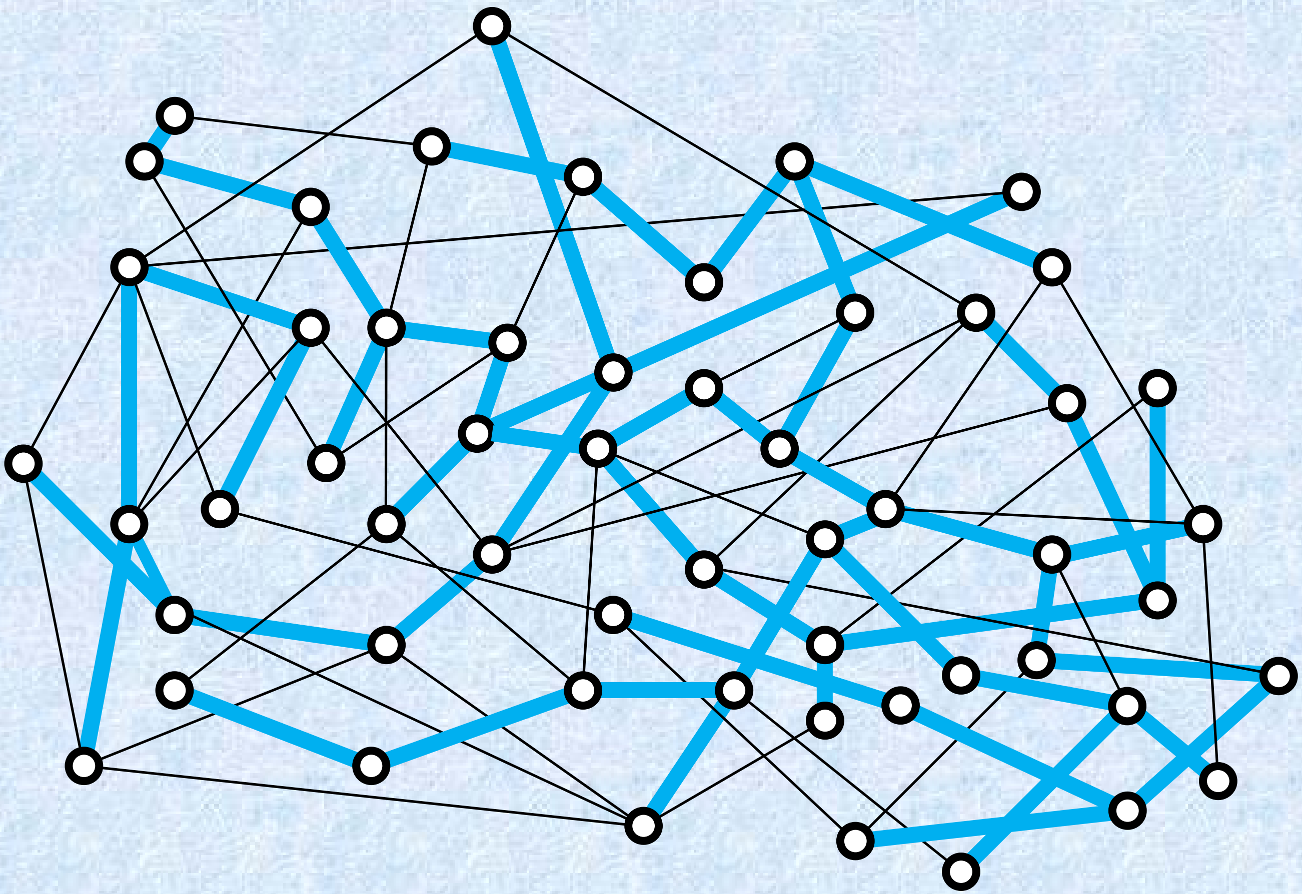


Minimum spanning tree

Minimum total cost (weight) of selected edges which connect all nodes in the graph. The selected edges form a tree.

Easy problem

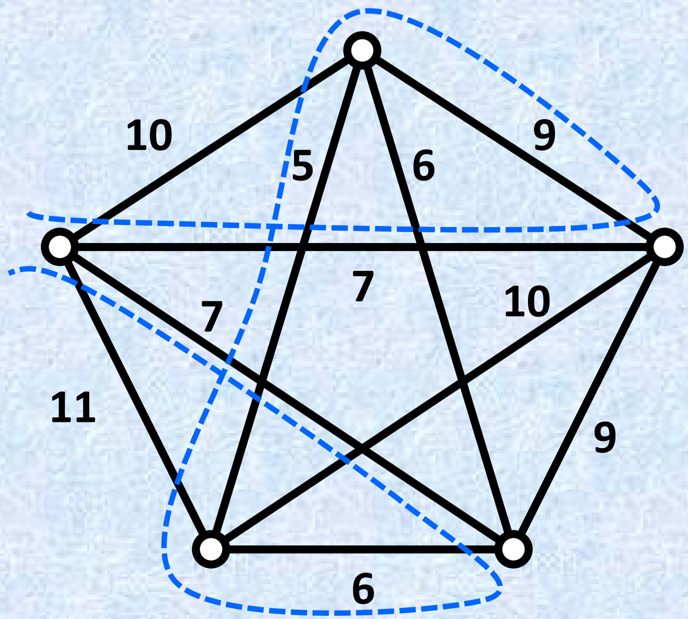
Here, the cost of an edge is proportional to its length (prefer shortest edges possible)



Travelling salesman problem (TSP)

Traverse a complete weighted graph, visit each node once and pay the minimum price for the journey = sum of costs of all visited edges.

Hard problem



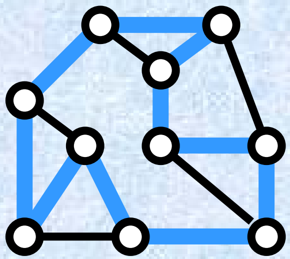
Hamilton path

Is there a path in the graph which contains each node (exactly once) ?

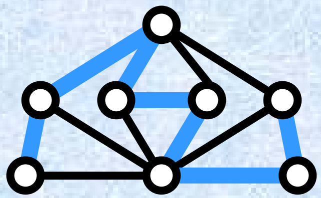
Hamilton cycle

Is there a cycle in the graph which contains each node?

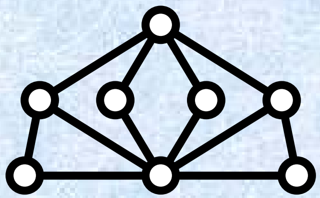
Hard problem



Both Hamilton path and Hamilton cycle exist.



Only Hamilton path exists. There is no Hamilton cycle.



Neither a Hamilton path nor a Hamilton cycle exists.

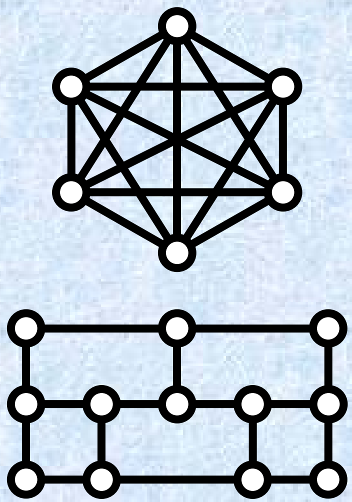
Euler trail

A trail that visits every edge exactly once (allowing for revisiting vertices)?

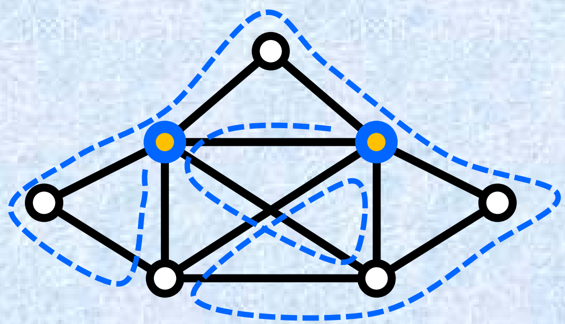
Ex: Can a postman walk through each street in their region exactly once?

Easy problem **Graph must be connected and it must contain at most 2 nodes of odd degree.**

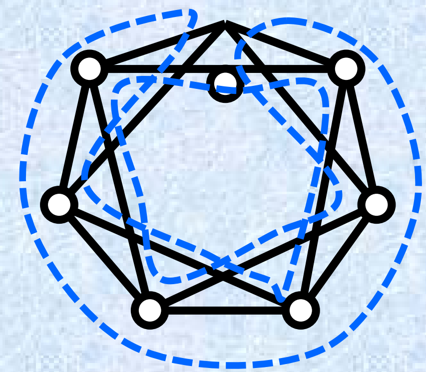
Algorithm: Hierholzer's O(|E|)



Euler trail does not exist, there are > 2 nodes with odd degree



The trail starts and ends in the nodes with odd degree



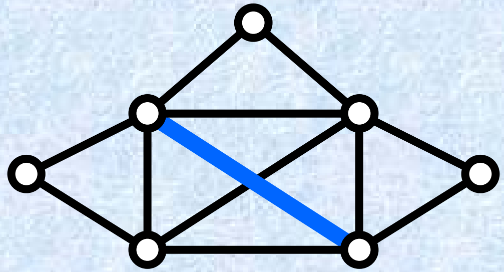
The trail is closed, all node degrees are even

Planar graph

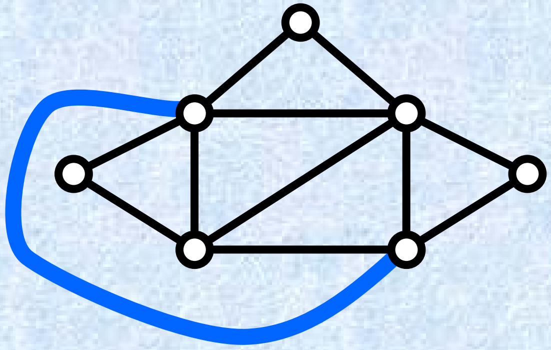
Can the graph be drawn in a plane wihout crossing its edges?

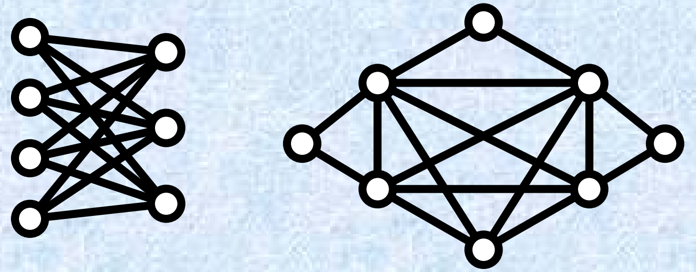
Easy question (however, little bit more advanced) ☺

Algorithms: Hopcroft and Tarjan, O(|V|)
Boyer and Myrvold, O(|V|)



The graph is planar, the blue edge can be drawn differently:

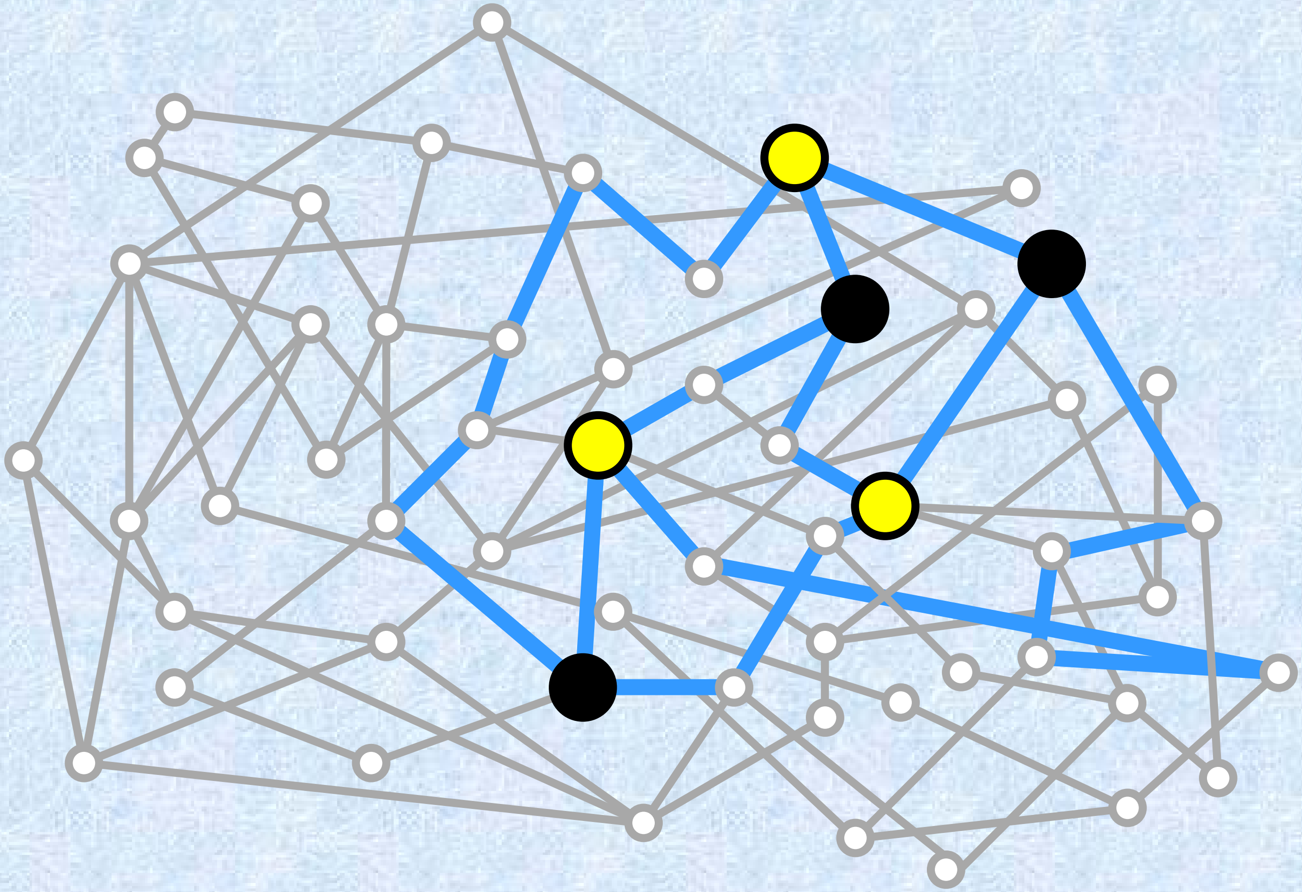




Not planar.
Non planar graphs "contain" either
a *complete graph on 5 nodes*
or a *complete bipartite graph on 3 and 3 nodes*.
The planar graphs do not "contain" them.

Planar graph

Can the graph be drawn in a plane wihout crossing its edges?



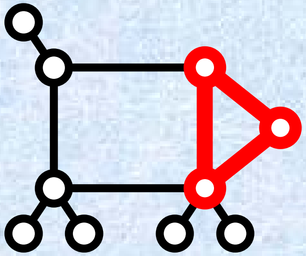
It is impossible here. Each black node is connected to each yellow node by a separate path. It is the case of a complete bipartite graph with partitions of size 3 and 3 (so called $K_{3,3}$). That graph cannot be drawn in the plane without edges crossing(s).

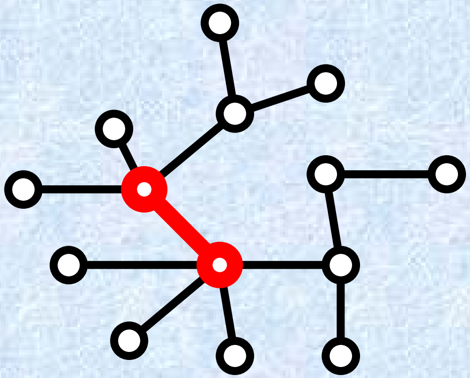
Clique number

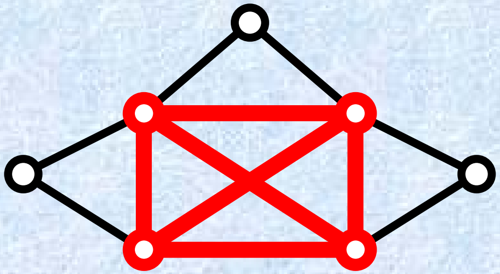
The size of the maximal clique, that is, of a subgraph which is complete, that is, of the subgraph where each node is connected to each other node.

Ex. Choose a largest group of your friends in which everybody knows each other.

Hard problem



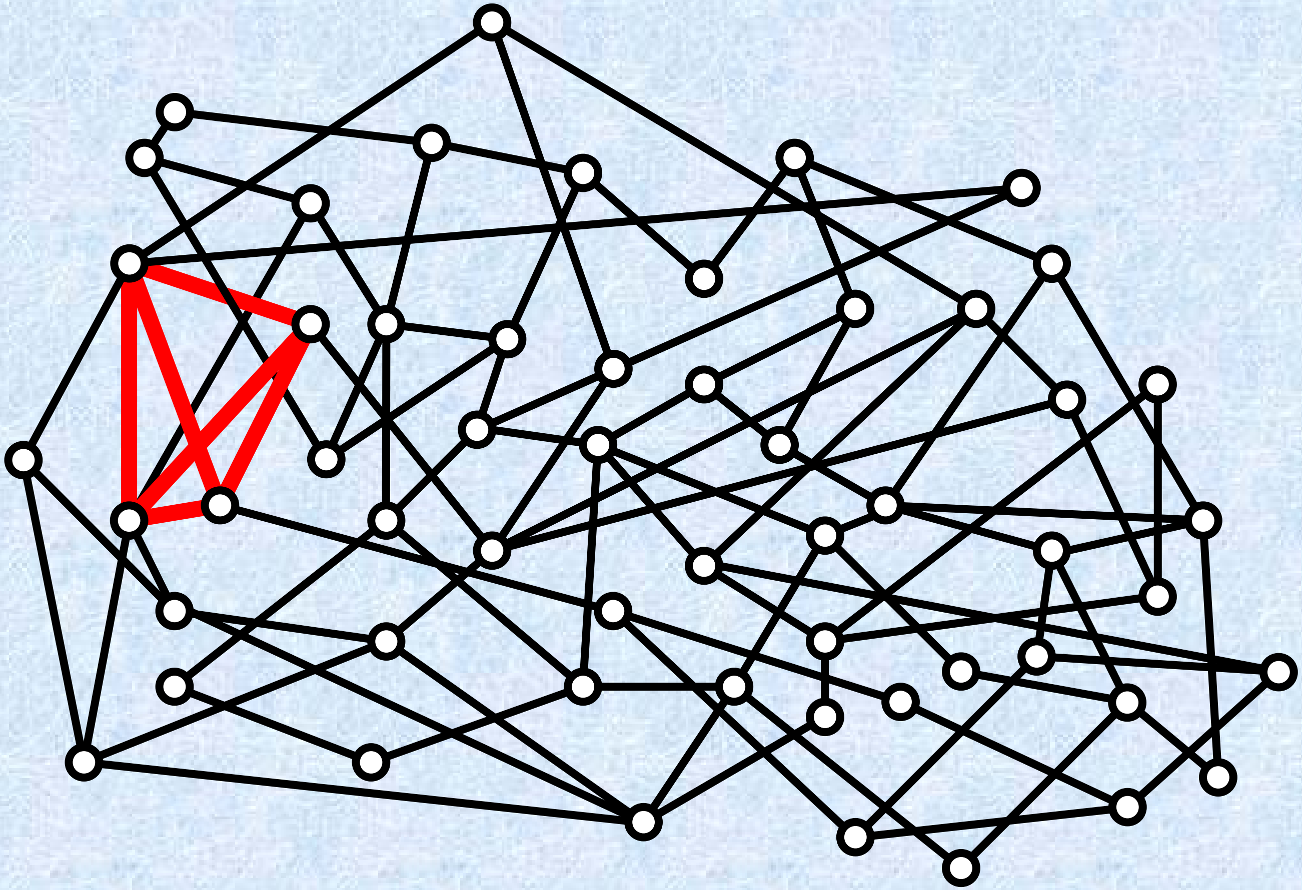




Clique number of all trees is 2.
(Rather obviously)

Clique number

The size of the maximal clique, that is, of a subraph which is complete,
that is, of the subgraph where each node is connected to each other node.

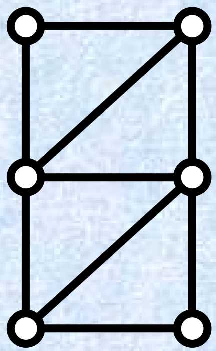


Clique of size 5 (or bigger) is not in the graph. To verify it mechanically, it is enough to check neighbour relations in all 5-element subsets of nodes. The number of those subsets is COMB(55, 5) = 3 478 761.

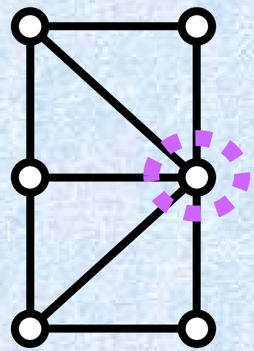
Graph isomorphism

Is the structure of two graphs identical? In other words,
can one graph be drawn in such way that it looks exactly as the other one?

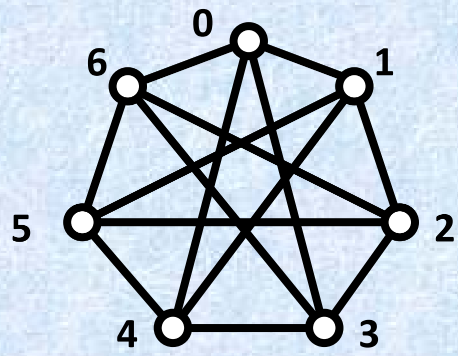
It is not know if this is a hard problem or an easy problem.



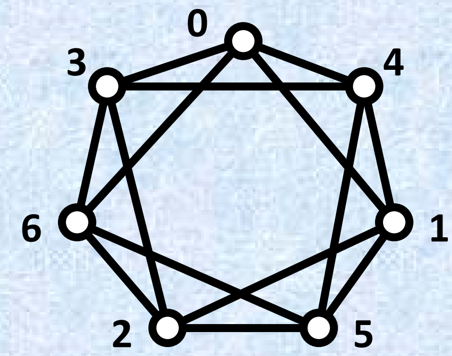
A



B



C



D

**A and B are not isomorphic,
right central node in B has degree 5,
there is no analogous node
anywhere in A. The structure of
A and B must be different.**

**C and D are isomorphic,
the nodes with the same labels
correspond to each other,
the edges in both C and D connect
the nodes with the same labels**

Partial recapitulation of the jungle of graph problems and their complexities

Easy problem

Connectivity?

Shortest path?

Min. spanning tree?

Euler trail?

Planarity?

"It depends... "

Colorability?

1,2 colors — easy

3 or more colors — hard

Isomorphism?

Trees, ciculants... — easy

regular graphs... — hard

etc...

Longest path?

DAG, tree — easy

general graph — hard

Hard problem

Travelling salesman?

Independence?

Dominancy?

Hamiltonicity?

Clique number?

Many more questions ... ? Again, "it depends". There is no definite cookbook for determining the difficulty of a problem.

	Shortest paths			
Edge weights	**Tree**	**DAG**	**Sparse graph with cycles**	**Dense graph with cycles**
Non-negative **or** **no weights, like all weights == 1**	**Directed or undirected** BFS, Θ(N+E) = Θ(N) *trivial problem*	**Only directed** topological sort and DP, Θ(N+E)	**Directed or undirected** Dijkstra **with** priority queue, Θ((N+E) logN)	**Directed or undirected** Dijkstra **without** priority queue, $\Theta(N^2)$
Some weights negative, but no neg. cycles! (conservative weights)			**Directed:** Bellman-Ford, Θ(N*E) **Undirected:** Transform to problem "Minimum Weight T-Join", $O(N^3)$, slightly advanced, see e.g. [KorteVygen, p.278].	
negative cycle exists	~~*undefined*~~	~~*Undefined*~~	**Directed or undirected** NP-hard when shortest path should be without cycles, otherwise undefined.	

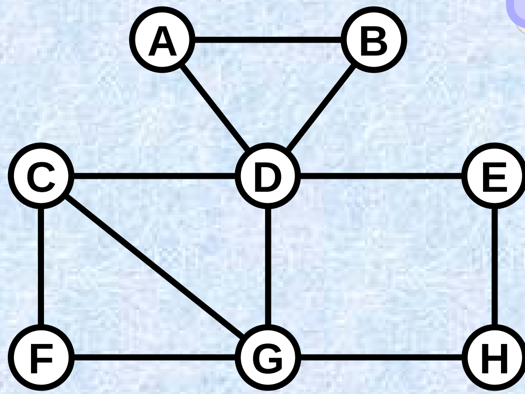
Bernhard Korte, Jens Vygen: *Combinatorial Optimization, Theory and Algorithms*, 3rd edition, Springer-Verlag, 2006.

	LONGEST PATHS		
	Tree	**DAG**	**Graph with cycles**
Any edge weights **or** **no weights** **or** **all weights equal**	**Directed or undirected** BFS, Θ(N+E) = Θ(N) *trivial problem*	**Only directed** topological sort and DP, Θ(N+E)	**Directed or undirected** NP-hard

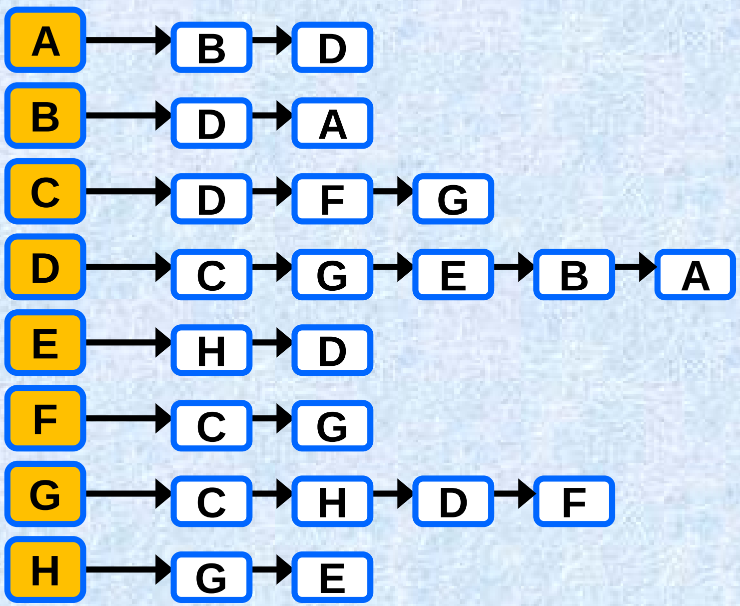
Graph most ususal representations

Graph most ususal representations

Undirected graph



Linked list representation

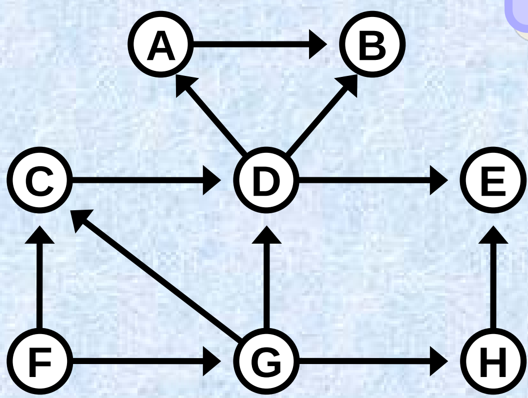


Adjacency matrix

	A	B	C	D	E	F	G	H
A	0	1	0	1	0	0	0	0
B	1	0	0	1	0	0	0	0
C	0	0	0	1	0	1	1	0
D	1	1	1	0	1	0	1	0
E	0	0	0	1	0	0	0	1
F	0	0	1	0	0	0	1	0
G	0	0	1	1	0	1	0	1
H	0	0	0	0	1	0	1	0

Graph most ususal representations

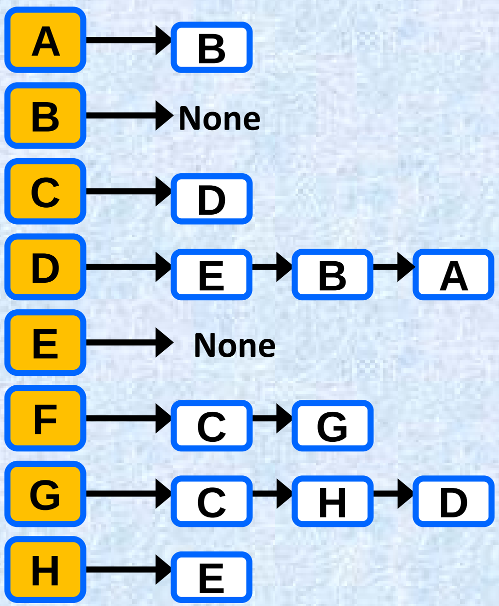
Directed graph



Adjacency matrix

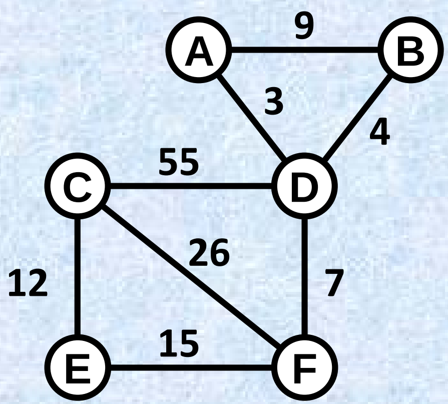
	A	B	C	D	E	F	G	H
A	0	1	0	0	0	0	0	0
B	0	0	0	0	0	0	0	0
C	0	0	0	1	0	0	0	0
D	1	1	0	0	1	0	0	0
E	0	0	0	0	0	0	0	0
F	0	0	1	0	0	0	1	0
G	0	0	1	1	0	0	0	1
H	0	0	0	0	1	0	0	0

Linked list representation

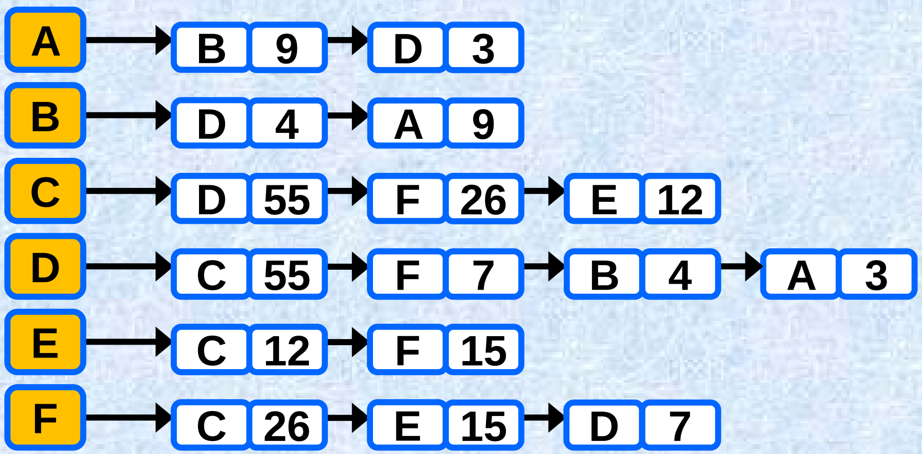


Graph most ususal representations

Undirected weighted graph



Linked list representation



Weight (cost) matrix

	A	B	C	D	E	F
A	0	9	0	3	0	0
B	9	0	0	4	0	0
C	0	0	0	55	12	26
D	3	4	55	0	0	7
E	0	0	12	0	0	15
F	0	0	26	7	15	0

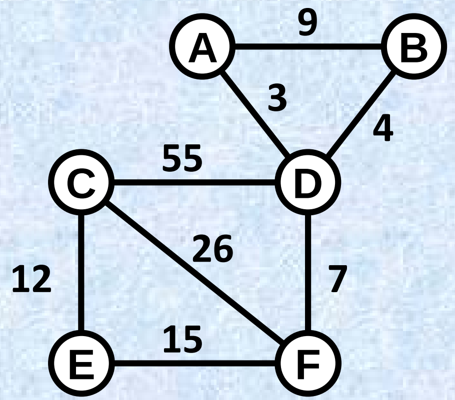
Graph most ususal representations

Linked list/ array representation

	0	1	2	3
A	B	D		
B	D	A		
C	D	F	E	
D	C	F	B	A
E	C	F		
F	C	E	D	

A	9	3		
B	4	9		
C	55	26	12	
D	55	7	4	3
E	12	15		
F	26	15	7	

Undirected weighted graph

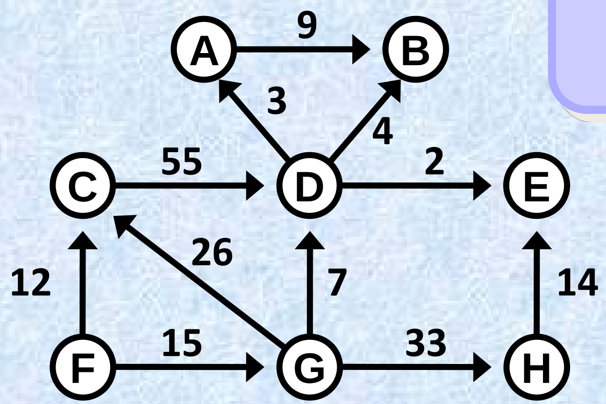


The weights of edges are at the same index in the second list.

\+ Pro: Simpler object or even no objects at all in the arrays.

\- Con: Keeping lists in sync needs more care and caution in the code.

Graph most ususal representations



Directed weighted graph

The representation is usually a more or less obvious combination of the methods in the previous cases -- Weight matrix or linked list.