Network Properties

Network Application Diagnostics
B2M32DSA

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Outline

1. Graph Matrices
   - Linear Algebra Reminder
   - Network Matrices
2. Centrality Measures
   - Path Based Centralities
   - Spectral Centralities
   - Example

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Algebra

- δ_{ij} is the Kronecker delta, which is 1 if $i = j$ and 0 otherwise.
- A **field** (CZ pole, komutativní těleso)is a set on which are defined addition, subtraction, multiplication, and division satisfying the field axioms (commutativity, associativity, a unit).
- $\mathbf{1}$ is the vector $(1, 1, 1, \dots)$.
- The **complex conjugate** (CZ komplexně sdružené číslo) of the complex number $z = x + iy$ is defined to by $\bar{z} = z^* = x - iy$.

Matrix [Lay12, GL13]

- $[\dots]_{ij}$ denotes (i, j) element of a matrix
- The **conjugate** of a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n \times m}$ is the matrix $\bar{\mathbf{A}} = (\bar{a}_{ij}) \in \mathbb{C}^{n \times m}$.
- The **trace** of an $n \times n$ ("n by n") square matrix $\mathbf{A}$ is

$$\text{Tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn} \quad (1)$$

$$\text{Tr}(\mathbf{A} + \mathbf{B}) = \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B}) \quad (2)$$

$$\text{Tr}(c\mathbf{A}) = c\text{Tr}(\mathbf{A}) \quad (3)$$

$$\text{Tr}(\mathbf{A}) = \text{Tr}(\mathbf{A}^T) \quad (4)$$

$$\text{Tr}(\mathbf{AB}) = \text{Tr}(\mathbf{BA}) \quad (5)$$

Matrix Transposition [Wat02, Lay12, GL13]

- The **transpose** of a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ ($\mathbb{R}^{n\times m} \to \mathbb{R}^{m\times n}$): $[\mathbf{A}^T]_{ij} = [\mathbf{A}]_{ji}$.
- Let $\mathbf{A}$ and $\mathbf{B}$ denote matrices whose sizes are appropriate for the following sums and products, let r denote any scalar, then
  - $(\mathbf{A}^T)^T = \mathbf{A}$
  - $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
  - $(r\mathbf{A})^T = r\mathbf{A}^T$
  - $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$
- The **conjugate transpose** of a matrix $\mathbf{A} \in \mathbb{C}^{n\times m}$: $[\mathbf{A}^*]_{ij} = [\bar{\mathbf{A}}]_{ji}$.
- The square matrix $\mathbf{A}$ is **Hermitian** if $\mathbf{A}^* = \mathbf{A} = \mathbf{A}^H$ and **skew-Hermitian** if $\mathbf{A}^* = -\mathbf{A}$.

Orthogonality [Wat02, GL13]

- A set of vectors $\{x_1, \ldots, x_p\}$ in $\mathbb{R}^n$ is **orthogonal** if $x_i^T x_j = 0$ whenever $i \neq j$ and **orthonormal** if $x_i^T x_j = \delta_{ij}$.
- A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be **orthogonal** if $\mathbf{A}^T \mathbf{A} = \mathbf{I}$.
- A matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is said to be **unitary** if $\mathbf{A}^* \mathbf{A} = \mathbf{I}$.

Matrix Inversion [GL13]

- If $\mathbf{A}$ and $\mathbf{X}$ are in $\mathbb{R}^{n \times n}$ and satisfy $\mathbf{AX} = \mathbf{I}$, then $\mathbf{X}$ is the **inverse** of $\mathbf{A}$ and is denoted by $\mathbf{A}^{-1}$.
  - $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
  - $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \equiv \mathbf{A}^{-T}$

Matrix Eigenvalues [GL13]

- The **eigenvalues** of $\mathbf{A} \in \mathbb{C}^{n \times n}$ are zeros of the **characteristic polynomial** $p(x) = det(\mathbf{A} - x\mathbf{I})$.
- Every $n \times n$ matrix has n eigenvalues.
- We denote the set of $\mathbf{A}$'s eigenvalues by

$$\lambda(\mathbf{A}) = \{x : det(\mathbf{A} - x\mathbf{I}) = 0\}$$

$$\lambda_{\max}(\mathbf{A}) = \max(\lambda(\mathbf{A})) \qquad \lambda_{\min}(\mathbf{A}) = \min(\lambda(\mathbf{A}))$$

- The **eigenvalue equation** expressed as the matrix multiplication

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

  - Applying the matrix $\mathbf{A}$ to the eigenvector $\mathbf{v}$ only scales the eigenvector by the scalar value λ.
- Symmetry of a matrix $\mathbf{A}$ guarantees that all of its eigenvalues are real and that there is an orthonormal basis of eigenvectors.
- Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ with eigenvalues λ and eigenvectors $\mathbf{v}$. Then $\mathbf{A}^k$ has eigenvalues λ^k and eigenvectors $\mathbf{v}$ for any positive integer k.

Schur Decomposition [GL13]

Theorem 1 (Symmetric Schur Decomposition, Theorem 8.1.1 [GL13], p.440)

If $\mathbf{A} \in \mathbb{R}^{n \times n}$ *is symmetric, then there exists a real orthogonal* $\mathbf{Q}$ *such that*

$$\mathbf{Q}^T \mathbf{A} \mathbf{Q} = \mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_n).$$

Moreover, for $k = 1 : n$, $\mathbf{A}\mathbf{Q}(:, k) = \lambda_k \mathbf{Q}(:, k)$.

Theorem 2 (Schur Decomposition, Theorem 7.1.3 [GL13], p.351)

If $\mathbf{A} \in \mathbb{C}^{n \times n}$, *then there exists a unitary* $\mathbf{Q} \in \mathbb{C}^{n \times n}$ *such that*

$$\mathbf{Q}^H \mathbf{A} \mathbf{Q} = \mathbf{T} = \mathbf{\Lambda} + \mathbf{N}$$

where $\mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_n)$ *and* $\mathbf{N} \in \mathbb{C}^{n \times n}$ *is strictly upper triangular.*

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Adjacency Matrix [New10, EK10]

- The **adjacency matrix** $\mathbf{A}$ of a *simple* graph is the $N \times N$ matrix with element A_{ij} such that

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices } j \text{ and } i, \\ 0 & \text{otherwise} \end{cases}$$

- The adjacency matrix of a *directed* network has matrix elements

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from } j \text{ to } i, \\ 0 & \text{otherwise} \end{cases}$$

Cocitation Matrix [New10]

- Convenient to turn a directed network into an undirected one for the purposes of analysis
- The **cocitation** of two vertices i and j in a directed network is the number of vertices that have outgoing edges pointing to both i and j.
  - The cocitation of two papers is the number of other papers that cite both.
  - $A_{ik}A_{jk} = 1$ if i and j are both cited by k and zero otherwise.
- The cocitations C_{ij} of i and j is

$$C_{ij} = \sum_{k=1}^{N} A_{ik}A_{jk} = \sum_{k=1}^{N} A_{ik}A_{kj}^{T}$$

- The **cocitation matrix** $\mathbf{C}$ is the $N \times N$ matrix with elements C_{ij}, i.e.

$$\mathbf{C} = \mathbf{A}\mathbf{A}^T$$

- $\mathbf{C}$ is a symmetric matrix: $\mathbf{C}^T = (\mathbf{A}\mathbf{A}^T)^T = \mathbf{A}\mathbf{A}^T = \mathbf{C}$

Bibliographic Coupling [New10]

- The **bibliographic coupling** of two vertices in a directed networkis the number of other vertices to which both point.
  - For instance in a citation network: the bibliographic coupling of two papers i and j is the number of other papers that are cited by both i and j.
  - $A_{ki}A_{kj} = 1$ if i and j both cite k and zero otherwise.
- The bibliographic coupling B_{ij} of i and j is

$$B_{ij} = \sum_{k=1}^{N} A_{ki}A_{kj} = \sum_{k=1}^{N} A_{ik}^{T}A_{kj}$$

- The **bibliographic coupling matrix** $\mathbf{B}$ is the $n \times n$ matrix with elements B_{ij}, i.e.

$$\mathbf{B} = \mathbf{A}^T\mathbf{A}$$

- $\mathbf{B}$ is a symmetric matrix: $\mathbf{B}^T = (\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A} = \mathbf{B}$

Bi-adjacency Matrix [New10, BJP17]

Bipartite networks

- also called **two-mode** networks in SNA [New10]
- $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$
- movies $\times$ actors
- articles $\times$ authors
- timestamps $\times$ active Wifi access points (AP)
- people $\times$ groups

- Let $N_1 = |V_1|$ and $N_2 = |V_2|$,
  then the **bi-adjacency** matrix $\mathbf{B}$ [BJP17] is $N_1 \times N_2$ matrix having elements

$$B_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices } n_i \in V_1 \text{ and } n_j \in V_2, \\ 0 & \text{otherwise} \end{cases}$$

- Also called **incidence matrix** [New10], **bipartite adjacency matrix** [BM08]

Adjacency and Bi-adjacency Matrix [New10, BJP17]

$$\mathbf{A} = \begin{pmatrix} \emptyset_{|V_1|} & \mathbf{B} \\ \mathbf{B}^T & \emptyset_{|V_2|} \end{pmatrix}$$

Bipartite network and its bi-adjacency Matrix

TODO

Incidence Matrix [Die05, New10]

- The **incidence matrix** $\mathbf{B}$ by [Die05] of a *simple undirected* graph $G(V,E)$ with N vertices $V = \{v_1, \ldots, v_N\}$ and M edges $E = \{e_1, \ldots, e_M\}$ over the 2-element field $F_2 = \{0, 1\}$ is defined as the $N \times M$ matrix with elements B_{ij} such that

$$B_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise} \end{cases}$$

- The **edge incidence matrix** by Newman [New10] of a *simple undirected* graph $G(V,E)$ with N vertices and M edges is an $M \times N$ matrix $\mathbf{B}$ with elements B_{ij}

$$B_{ij} = \begin{cases} +1 & \text{if end 1 of edge } i \text{ is attached to vertex } j, \\ -1 & \text{if end 2 of edge } i \text{ is attached to vertex } j, \\ 0 & \text{otherwise} \end{cases}$$

  - Each edge has two arbitrarily designated ends, *end 1* and *end 2*.
  - Each row of the matrix has exactly one $+1$ and one -1 element.

Projection [New10, BJP17]

- A possible way how to analyze bipartite graphs using simple graph methods.
- Significant information on the given network might be lost.

Definition 1 (Based on Definition 3 [BJP17], p.3)

Let $G(V_1, V_2, E)$ be a bipartite graph. The **one-mode projection** of the bipartite graph G for the vertex V_i with respect to the vertex set V_j, $i, j \in \{1, 2\}$, $i \neq j$ is the unipartite (one-mode) network $G'(V_i, E')$ where $V(G') = U$ and $uv \in E(G')$ if $N(u) \cap N(v) \neq \emptyset$.

Projection of a bipartite network - items and groups

TODO

Projection Properties I [New10]

- Let $\mathbf{B}$ be a bi-adjacency matrix of $G(V_1, V_2, E)$, then the total number $P_{ij}^{(1)}$ of vertexes $v \in V_2$ to which both $i, j \in V_1$ belong is

$$P_{ij}^{(1)} = \sum_{k=1}^{|V_2|} B_{ik} B_{jk} = \sum_{k=1}^{|V_2|} B_{ik} B_{kj}^T$$

  - The product $B_{ik}B_{jk}$ will be 1 if and only if i and j are both linked to the same vertex k from the other vertex set
- Example: relations of items and their groups
- In matrix form

$$\mathbf{P}^{(1)} = \mathbf{B}\mathbf{B}^T$$

Projection Properties II [New10]

- $P_{ii}^{(1)}$ is the number of vertexes $j \in V_2$ to which $i \in V_1$ is linked

$$P_{ij}^{(1)} = \sum_{k=1}^{|V_2|} B_{ik}^2 = \sum_{k=1}^{|V_2|} B_{ik}$$

  - assuming $B_{ik} \in \{0, 1\}$
- The other one-mode projection onto V_2

$$\mathbf{P}^{(2)} = \mathbf{B}^T\mathbf{B}$$

Undirected Graph - Node Degree [New10]

- The **degree** of a vertex in a undirected graph

$$k_i = \sum_{j=1}^{N} A_{ij}$$

- The number of ends of edges

$$2M = \sum_{i=1}^{N} k_i$$

- The number of edges

$$M = \frac{1}{2} \sum_{i=1}^{N} k_i = \frac{1}{2} \sum_{ij} A_{ij}$$

Undirected Graph - Density [New10]

- The **mean degree** c of a vertex in a undirected graph

$$c = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{2M}{N}$$

- The maximum possible number of edges in a simple graph

$$\binom{N}{2} = \frac{1}{2}N(N-1)$$

- The **connectance** or **density** ρ of a graph is the fraction of edges that are actually present $(0 \leq \rho \leq 1)$.

$$\rho = \frac{M}{\binom{N}{2}} = \frac{2M}{N(N-1)} = \frac{c}{N-1}$$

Directed Graph - Vertex Degree [New10]

- The **in-degree** k_i^{in} and **out-degree** k_j^{out} of a vertex in a undirected graph

$$k_i^{\text{in}} = \sum_{j=1}^{N} A_{ij}, \quad k_j^{\text{out}} = \sum_{i=1}^{N} A_{ij}$$

- The number of edges

$$M = \sum_{i=1}^{N} k_i^{\text{in}} = \sum_{j=1}^{N} k_j^{\text{out}} = \sum_{ij} A_{ij}$$

- The **mean in-degree** c_{in} and the **mean out-degree** c_{out} of a vertex in a undirected graph are equal:

$$c_{\text{in}} = \frac{1}{N}\sum_{i=1}^{N} k_i^{\text{in}} = \frac{1}{N}\sum_{j=1}^{N} k_j^{\text{out}} = c_{\text{out}} = c = \frac{M}{N}$$

Paths in Simple Graph [New10]

- The element A_{ij} is 1 if there is an edge from i to j, and 0 otherwise in simple graphs.
- The product $A_{ik}A_{kj}$ is 1 if there is a path of length 2 from j to i via k, and 0 otherwise.
- The total number $N_{ij}^{(2)}$ of paths of length two from j to i via any other vertex is

$$N_{ij}^{(2)} = \sum_{k=1}^{N} A_{ik}A_{kj} = [\mathbf{A}^2]_{ij}$$

- Paths of length three from j to i via l and k in that order

$$N_{ij}^{(3)} = \sum_{k=1}^{N} A_{ik}A_{k\ell}A_{\ell j} = [\mathbf{A}^3]_{ij}$$

- Paths of an arbitrary length r

$$N_{ij}^{(r)} = [\mathbf{A}^r]_{ij}$$

Cycles in Simple Graph [New10]

- The number of paths of length r that start and end at the same vertex i is $[\mathbf{A}^r]_{ii}$.
- The total number L_r of cycles ("loops") of length r anywhere in a network is (the sum over all possible starting vertexes i)

$$L_r = \sum_{i=1}^{N} [\mathbf{A}^r]_{ii} = \mathrm{Tr}\mathbf{A}^r.$$

- The loop $1 \to 2 \to 3 \to 1$ is considered different from the loop $2 \to 3 \to 1 \to 2$.
- The loops $1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 2 \to 1$ traversed in opposite directions are distinct, too.

Cycles in Simple Graph and Eigenvalues [New10]

- **Undirected graph**
  - The adjacency matrix $\mathbf{A}$ is symmetric, i.e. $\mathbf{A} = \mathbf{Q}\mathbf{K}\mathbf{Q}^T$, where $\mathbf{Q}$ is the orthogonal matrix of eigenvectors and $\mathbf{K}$ is the diagonal matrix of eigenvalues κ_i of $\mathbf{A}$.
  - $\mathbf{A}^r = (\mathbf{Q}\mathbf{K}\mathbf{Q}^T)^r = \mathbf{Q}\mathbf{K}^r\mathbf{Q}^T$
  - $L_r = \mathrm{Tr}\mathbf{A}^r = \mathrm{Tr}(\mathbf{Q}\mathbf{K}^r\mathbf{Q}^T) = \mathrm{Tr}(\mathbf{Q}^T\mathbf{Q}\mathbf{K}^r) = \mathrm{Tr}\mathbf{K}^r = \sum_i \kappa_i^r$
- **Directed networks**
  - Every real matrix can be written in the form $\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$, where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{T}$ is an upper triangular matrix using the *Schur decomposition*.
  - Since $\mathbf{T}$ is triangular, its diagonal elements are its eigenvalues.
  - The eigenvalues are the same as the eigenvalues of $\mathbf{A}$.

$$\mathbf{A}\mathbf{x} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T\mathbf{x} = \kappa\mathbf{x} \qquad \cdots \times \mathbf{Q}^T \tag{6}$$

$$\mathbf{T}\mathbf{Q}^T\mathbf{x} = \kappa\mathbf{Q}^T\mathbf{x} \tag{7}$$

$$\tag{8}$$

  - $L_r = \mathrm{Tr}\mathbf{A}^r = \mathrm{Tr}(\mathbf{Q}\mathbf{T}^r\mathbf{Q}^T) = \mathrm{Tr}(\mathbf{Q}^T\mathbf{Q}\mathbf{T}^r) = \mathrm{Tr}\mathbf{T}^r = \sum_i \kappa_i^r$

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Centrality Measures / Ranking [BE06, Weh13]

Measuring the importance/prominence of a node within the network

- Degree Centrality (Node Activity)
- Betweenness Centrality (Intermediate Position)
- Closeness Centrality (Distance to other nodes)
- Eigenvector Centrality (Important nodes have important friends)
- Power Centrality (Close to hubs)
- Page Rank

Evaluation of the location actors in the network

- Insight into various roles and groupings in a network
- Connectors, mavens, leaders, bridges, isolates, broker, hubs
- Where are the clusters and who is in them,
- Who is in the core of the network? Who is on the periphery?
- What is a single point of failure?

Degree Centrality [Fre79, BE06, Weh13]

What is the degree of an actor? How active is an actor?

Degree centrality

is a count of the number of edges incident upon a given vertex.

Degree centrality for actor i

$$c_i^d = \sum_j a_{ij} = \mathbf{A1}$$

- where $\mathbf{A}$ is the adjacency matrix
- $\mathbf{1}$ is a vector of 1 with size N.

Normalized degree centrality for actor i

$$c_i'^d = \frac{\sum_j a_{ij}}{N-1} = \frac{\mathbf{A1}}{N-1}$$

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Examples of degree centrality [Weh13]

Examples for degree centrality c_i and normalized degree centrality c'_i:

Star

$c_1^d = 4 \quad {c'}_1^d = 1$
$c_2^d = 1 \quad {c'}_2^d = 0.25$
$c_3^d = 1 \quad {c'}_3^d = 0.25$
$c_4^d = 1 \quad {c'}_4^d = 0.25$
$c_5^d = 1 \quad {c'}_5^d = 0.25$

Line

$c_1^d = 2 \quad {c'}_1^d = 0.5$
$c_2^d = 2 \quad {c'}_2^d = 0.5$
$c_3^d = 2 \quad {c'}_3^d = 0.5$
$c_4^d = 1 \quad {c'}_4^d = 0.25$
$c_5^d = 1 \quad {c'}_5^d = 0.25$

Circle

$c_1^d = 2 \quad {c'}_1^d = 0.5$
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(all actors identical)

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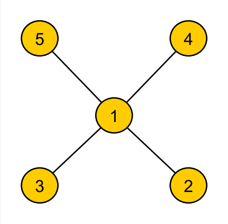

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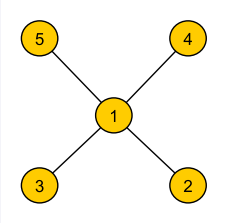

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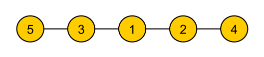

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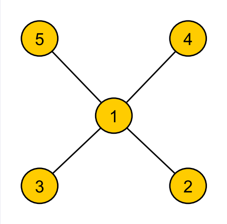

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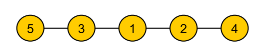

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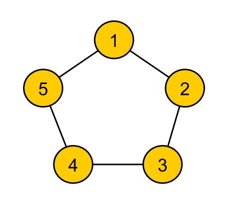
$$c_3^d = 2 \quad c'^d_3 = 0.5$$
$$c_4^d = 1 \quad c'^d_4 = 0.25$$
$$c_5^d = 1 \quad c'^d_5 = 0.25$$

Circle



$$c_1^d = 2 \quad c'^d_1 = 0.5$$
$$c_2^d = 2 \quad c'^d_2 = 0.5$$
$$c_3^d = 2 \quad c'^d_3 = 0.5$$
$$c_4^d = 2 \quad c'^d_4 = 0.5$$
$$c_5^d = 2 \quad c'^d_5 = 0.5$$

(all actors identical)

Closeness centrality [Fre79, Dod09]

- Idea: Nodes are more central if they can reach other nodes 'easily.'
- Measures average shortest path from a node to all other nodes.
- **Closeness Centrality** for node i as

$$c_i^c = \frac{N-1}{\sum_{j,j\neq i}(\text{distance from } i \text{ to } j)}$$

- Range is 0 (no friends) to 1 (a single hub).

Meaning

- Unclear what the exact values of this measure tells us because of its ad-hocness.
- General problem with simple centrality measures: what do they exactly mean?
- Perhaps, at least, we obtain an ordering of nodes in terms of 'importance.'

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Betweenness centrality [Dod09]

- Betweenness centrality is based on shortest paths in a network.
- **Idea:** If the quickest way between any two nodes on a network disproportionately involves certain nodes, then they are ‘important’ in terms of global cohesion.
- For each node i, **count, over all pairs of nodes x and y, how many shortest paths pass through i.**
- Call frequency of shortest paths passing through node i the **betweenness** of i, B_i .
- Note: Exclude shortest paths between i and other nodes.
- Note: works for weighted and unweighted networks.
- Role played by shortest paths justified by small-world phenomenon (Milgram’s experiment).

Betweenness Centrality - Complexity [Dod09]

- Consider a network with N nodes and M edges (possibly weighted).
- *Computational goal:* Find $\binom{N}{2}$ **shortest paths** between all pairs of nodes.
- Traditionally **Floyd-Warshall** algorithm used.
- Computation time grows as $O(N^3)$.
- See also:
  1. **Dijkstra's algorithm** for finding the shortest path between two specific nodes, and
  2. **Johnson's algorithm** which outperforms Floyd-Warshall for sparse networks:
  $$O(MN + N^2 logN)$$
  .
- Newman (2001) and Brandes (2001) independently derived much faster algorithms.
- Computation times grow as:
  1. $O(MN)$ for unweighted graphs, and
  2. $O(MN + N^2 logN)$ for weighted graphs.

Shortest path between node i and all others [Dod09]

- Consider unweighted networks.
- Use **breadth-first search**:
  1. Start at node i, giving it a distance $d = 0$ from itself.
  2. Create a list of all of i's neighbors and label them being at a distance $d = 1$.
  3. Go through list of most recently visited nodes and find all of their neighbors.
  4. Exclude any nodes already assigned a distance.
  5. Increment distance d by 1.
  6. Label newly reached nodes as being at distance d.
  7. Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from i (former are 'predecessors' with respect to i's shortest path structure).
- Runs in $O(M)$ time and gives N shortest paths.
- Find all shortest paths in $O(MN)$ time
- Much, much better than naive estimate of $O(MN^2)$.

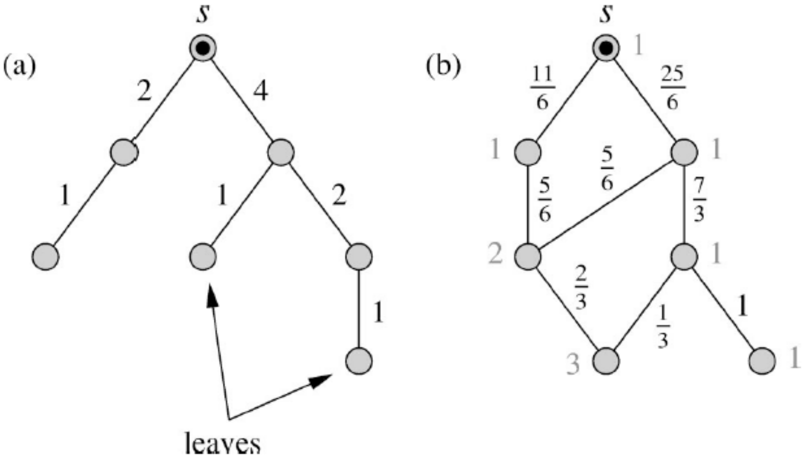
Newman's Betweenness algorithm [New01, Dod09]

1. Set all nodes to have a value $c_{ij} = 0, j = 1, \ldots, N$ (c for count).
2. Select one node i.
3. Find shortest paths to all other $N - 1$ nodes using breadth-first search.
4. Record $\#$ equal shortest paths reaching each node.
5. Move through nodes according to their distance from i, starting with the furthest.
6. Travel back towards i from each starting node j, along shortest path(s), adding 1 to every value of c_{ik} at each node k along the way.
7. Whenever more than one possibility exists, a portion according to total number of short paths coming through predecessors.
8. Exclude starting node j and i from increment.
9. Repeat steps 2-8 for every node i and obtain **betweenness** as $B_j = \sum_{i=1}^{N} c_{ij}$

Newman's Betweenness - notes [New01, Dod09]

- For a pure tree network, c_{ij} is the number of nodes beyond j from i's vantage point.
- For **edge betweenness**, use exact same algorithm but now
  1. j indexes edges, and
  2. we add one to each edge as we traverse it.
- For both algorithms, computation time grows as $O(MN)$ and space for $O(N+M)$ integers (N nodes, M arcs).
- Both bounds infeasible for large networks,
  where typically $N \approx 10^9$ and $M \approx 10^{11}$.
- For sparse networks with relatively small average degree, we have a fairly digestible time growth of $O(N^2)$.

Newman's Betweenness - examples [New01, Dod09]



Outline

1 Graph Matrices
- Linear Algebra Reminder
- Network Matrices

2 Centrality Measures
- Path Based Centralities
- Spectral Centralities
- Example

Important nodes have important friends [Dod09, New10]

- Define x_i as the "importance" of node i.
- *Idea*: x_i depends (somehow) on x_j if j is a neighbor of i.
- *Recursive*: importance is transmitted through a network.
- Simplest possibility is a linear combination:

$$x_i \propto \sum_j A_{ij} x_j$$

- Assume further that constant of proportionality, c, is independent of i.
- Above gives $\tilde{\mathbf{x}} = c\mathbf{A}\tilde{\mathbf{x}}$ or $\boxed{\mathbf{A}\tilde{\mathbf{x}} = c^{-1}\tilde{\mathbf{x}} = \lambda\tilde{\mathbf{x}}}$.
- Eigenvalue equation based on adjacency matrix:
  - The greatest eigenvalue and its related eigenvector fulfills only the additional requirement that all the entries in the eigenvector be positive (Perron-Frobenius theorem).
- **Eigenvalue centrality** of the vertex v in the network
  ... The v^{th} component of the related eigenvector

Eigenvalue Centrality - Iterative Approach [New10]

- An initial guess about the centrality x_i of each vertex i.
  - e.g. $x_i = 1$ for all i
- One step to calculate a better estimate x'_i

$$x'_i = \sum_j A_{ij} x_j \qquad \text{i.e. } \mathbf{x}' = \mathbf{A}\mathbf{x}$$

- Repeat t times: $\mathbf{x}(t) = \mathbf{A}^t \mathbf{x}(0)$
- Express $\mathbf{x}(0)$ as a linear combination of the eigenvectors v_i of $\mathbf{A}$: $\mathbf{x}(0) = \sum_i c_i \mathbf{v}_i$.

$$\mathbf{x}(t) = \mathbf{A}^t \sum_i c_i \mathbf{v}_i = \sum_i c_i \mathbf{A}^t \mathbf{v}_i = \sum_i c_i \kappa_i^t \mathbf{v}_i = \kappa_1^t \sum_i c_i [\frac{\kappa_i}{\kappa_1}]^t \mathbf{v}_i$$

  - κ_i are the eigenvalues of $\mathbf{A}$, κ_1 is the largest of them.
- Since $\kappa_i/\kappa_1 < 1$ for all $i \neq 1$, all terms in the sum other then the first decay exponentially as t becomes large: $\mathbf{x}(t) \to c_1 \kappa_1 \mathbf{v}_1$ as $t \to \infty$.

Eigenvalue Centrality - Properties [New10]

- **Eigenvalue centrality** by Bonacich in 1987 [Bon87]

$$\mathbf{A}\mathbf{x} = \kappa_1 \mathbf{x} \qquad x_i = \kappa_1^{-1} \sum_j A_{ij} x_j$$

- The centrality x_i of vertex i is proportional to the sum of the centralities of i's neighbors:
  - a vertex has many neighbors,
  - a vertex has important neighbors.
- The eigenvector centralities of all vertices are non-negative.
  - If $x_i(0) \geq 0$ and $A_{ij} \geq 0$ then $x_i(t) \geq 0$.
- Eigenvector centrality works well for *undirected* networks.
- Issues with *directed* networks
  - Asymmetric adjacency matrix has two sets of eigenvectors, left and right, i.e hence two leading eigenvectors.
  - In most cases the right eigenvector should be used
    - to prefer the case in which centralities are driven by vertices pointing to a given vertex (and not to which vertices the given vertex points to)
  - Zero x_i are propagated as zero $\implies$ strong components taken only.

Katz Centrality [Kat53]

- To resolve the issue with zero eigenvalue centralities x_i

Katz Centrality

- Proposed by Katz in 1953

$$\mathbf{C}_{\text{Katz}} = \alpha \mathbf{A} + \alpha^2 \mathbf{A}^2 + \dots + \alpha^k \mathbf{A}^k + \dots \tag{9}$$

$$\mathbf{C}_{\text{Katz}}(i) = \sum_{k=1}^{\infty} \sum_{j=1}^{N} \alpha^k [\mathbf{A}^k]_{ij} \tag{10}$$

- $\mathbf{C}_{\text{Katz}}(i)$ denotes Katz centrality of a node i.
- The attenuation factor α ... discounted paths (walks)
- A link in the distance k is attenuated by α^k.
- If $\alpha < 1/|\kappa_1|$, where κ_1 is the largest eigenvalue of $\mathbf{A}$, then

$$\vec{\mathbf{c}}_{\text{Katz}} = ((\mathbf{I} - \alpha \mathbf{A}^T)^{-1} - \mathbf{I})\mathbf{1}$$

Alpha Centrality [BL01, New10]

- Proposed by Bonacich in 2001 [BL01]
- A generalization of Katz centrality

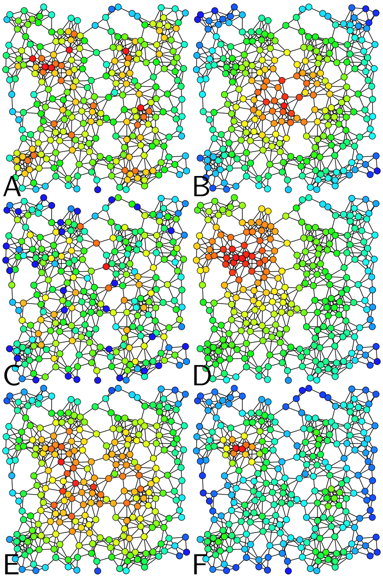
$$x_i = \alpha \sum_j A_{ij} x_j + \beta \qquad \mathbf{x} = \alpha \mathbf{A}\mathbf{x} + \beta \mathbf{1}$$

  where α and β are positive constants.
- Each vertex has a non-zero positive centrality because of small $\beta > 0$
- Rearranging for $\mathbf{x}$

$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A})^{-1} \cdot \mathbf{1} = (\mathbf{I} - \alpha\mathbf{A})^{-1} \cdot \mathbf{1}$$

  - using $\beta = 1$ to care about relative values of centralities only.
  - $\mathbf{C}_{\text{Alpha}} = \alpha^0 \mathbf{A}^0 + \mathbf{C}_{\text{Katz}} = \mathbf{I} + \mathbf{C}_{\text{Katz}}$
- Choice of a value of α
  - If $\alpha \to 0$, then all $x_i \to \beta = 1$
  - If $\alpha \to 1/\kappa_1$, then a divergence ... $\det(\mathbf{A} - \alpha^{-1}\mathbf{I}) = 0$

Centrality Measures - Importance of Nodes [Roc12]



- Low → middle → high values
- **A** Degree centrality,
  - Node Activity
- **B** Closeness centrality,
  - Distance to other nodes
- **C** Betweenness centrality,
  - Intermediate Position
- **D** Eigenvector centrality,
  - Important nodes have important friends
- **E** Katz centrality,
  - The relative influence of a node within a network
- **F** Alpha centrality
  - Important nodes have important friends for asymmetric relations

PageRank [BP98, BP12, New10]

- In some case, a high-centrality vertex should not distribute its centrality to other vertexes fully,
  - e.g. *Yahoo!* referencing a personal page.
- The centrality of a given vertex is distributed to its neighbors as an amount proportional to its centrality divided by its out-degree.

$$x_i = \alpha \sum_j A_{ij} \frac{x_j}{k_j^{\text{out}}} + \beta \qquad \mathbf{x} = \alpha \mathbf{A}\mathbf{D}^{-1}\mathbf{x} + \beta \mathbf{1}$$

  - If $k_j^{\text{out}} = 0$, then $A_{ij} = 0$ for all i.
  - In such cases, we set artificially $k_j^{\text{out}} = 1$ to avoid the problem with the term when zero is divided by zero. The result is a zero centrality contribution.
  - $\mathbf{D}$ is the diagonal matrix with elements $D_{ii} = \max(k_j^{\text{out}}, 1)$
- By rearranging and setting $\beta = 1$, and $\alpha < 1/|\kappa_1|$, $\kappa_1 = \lambda_{\max}(\mathbf{A})$

$$\mathbf{x} = \beta(\mathbf{I} - \alpha \mathbf{A}\mathbf{D}^{-1})^{-1} \cdot \mathbf{1} = \mathbf{D}(\mathbf{D} - \alpha \mathbf{A})^{-1} \cdot \mathbf{1}$$

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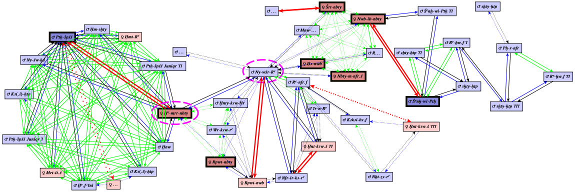
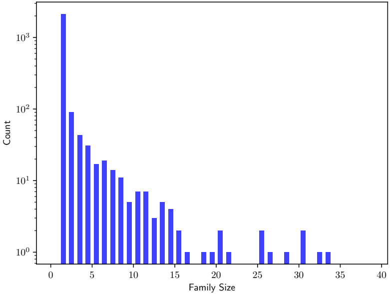
Egypt Data - Family Formation [?]

Ny-wśr-Rꜥ	0.647
Ḫꜥ-mrr-nbty	0.424
Nwb-ỉb-nbty	0.351
Śꜥnḫ-wỉ-Ptḥ	0.290
Rꜥ-ḫw.f I	0.180
Rꜥ-nfr.f	0.139
ꜣḫty-ḥtp III	0.139
Ptḥ-špśś	0.082
Pḥ-r-nfr III	0.048
Šrt-nbty I	0.048

People with
the top 10 highest betweenness





Extended family size distribution

Summary

- Linear algebra remainder
- Network matrices
- Centrality Measures
    - Path based centralities
    - Spectral centralities

Competencies

- Define adjacency matrix, cocitation matrix, and bibliographic coupling
- Define bi-adjacency matrix, incidence matrxi, edge incidence matrix
- Define one-mode projection and its relation to bi-adjacency matrix.
- Show how to compute degree of vertex, the number of edges, the mean degree, and graph density based on the adjacency matrix for undirected and directed graphs.
- Show how to compute number of paths and cycles based on the adjacency matrix.
- Define degree centrality.
- Define closenes centrality.
- Define betweenness centrality.
- Describe an algorithm for betweenness centrality computation.
- Define eigenvalue centrality.
- Define Katz centrality.
- Define PageRank index.

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