Simultaneous Localization and Mapping

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SLAM - Simultaneous Localization and Mapping

- The task of building a map while estimating the pose of the robot relative to this map.
- More difficult than separate localization or mapping.
- Chicken and egg problem: a map is needed to localize the robot and a pose estimate is needed to build a map.
- Given: robot control, sensing.
- Estimate: map (feature-based, grid), path of the robot
- State: $\langle x, y, \theta, \text{map} \rangle$
- Map:
  - Feature-based: $\langle l_1, l_2, \ldots, l_n \rangle$
  - Grid: $\langle c_{11}, c_{12}, \ldots c_{1n}, \ldots, c_{mn} \rangle$
Structure of the Landmark-based SLAM-Problem
Why is SLAM a hard problem?

- **SLAM**: robot path and map are both unknown!

- Robot path error correlates errors in the map.
Why is SLAM a hard problem?

- In the real world, the mapping between observations and landmarks is unknown.
- Picking wrong data associations can have catastrophic consequences.
- Pose error correlates data associations.
A data association is an assignment of observations to landmarks.

In general for $n$ measurements and $m$ landmarks there are $???$ possible associations.

Also called "assignment problem"
Data Association Problem

- A data association is an assignment of observations to landmarks.
- In general for $n$ measurements and $m$ landmarks there are \( \binom{m}{n} \) possible associations.
- Also called "assignment problem"
**SLAM**

**Full SLAM**
- Estimates entire path and map
- \( p(x_{1:t}, m|z_{1:t}, u_{1:t}) \)

**Online SLAM**
- Estimates most recent pose and map
- \[
p(x_t, m|z_{1:t}, u_{1:t}) = \int \int \cdots \int p(x_{1:t}, m|z_{1:t}, u_{1:t}) dx_1 dx_2 \ldots dx_{t-1}
\]
- Integrations typically done one at a time
Techniques for Generating Consistent Maps

• Scan matching
• EKF-SLAM
• Fast-SLAM
• Probabilistic mapping with a single map and a posterior about poses.
• Graph-SLAM
• Sparse Extended Information Filters (SEIFs)
• RAT-SLAM, . . .
Scan matching

- Maximization of the likelihood of the $i$-th pose and a map relative to the $(i-1)$-th position and a map.

$$\hat{x}_t = \arg \max_{x_t} \left\{ p(z_t|x_t, \hat{m}^{[t-1]}) p(x_t|u_{t-1}, \hat{x}_{t-1}) \right\}$$

- Calculate the map $\hat{m}^{[t]}$ according to “mapping with known poses” and observations.
EKF-SLAM

- Feature map (number of landmarks < 1000)
- State:
  \[ s_t = (x_t, m)^T = (x, y, \theta, l_{1,x}, l_{1,y}, l_{2,x}, l_{2,y}, \ldots, l_{n,x}, l_{n,y})^T \]
- Assuming knowledge of associations
- Only positive information is processed
- PDF represented with a high-dimensional Gaussian (3+2n)

\[
\text{Bel}(x_t, m_t) = \begin{pmatrix}
  x \\
  y \\
  \theta \\
  l_1 \\
  l_2 \\
  \vdots \\
  l_n
\end{pmatrix},
\begin{pmatrix}
  \sigma^2_x & \sigma_{xy} & \sigma_{x\theta} \\
  \sigma_{xy} & \sigma^2_y & \sigma_{y\theta} \\
  \sigma_{x\theta} & \sigma_{y\theta} & \sigma^2_\theta \\
  \sigma_{xl_1} & \sigma_{yl_1} & \sigma_{\theta l_1} \\
  \sigma_{xl_2} & \sigma_{yl_2} & \sigma_{\theta l_2} \\
  : & : & : \\
  \sigma_{xl_n} & \sigma_{yl_n} & \sigma_{\theta l_n}
\end{pmatrix}
\]

- Is matrix \( K \) (gain) sparse?
EKF-SLAM

- Feature map (number of landmarks < 1000)
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  \[ s_t = (x_t, m)^T = (x, y, \theta, l_{1,x}, l_{1,y}, l_{2,x}, l_{2,y}, \ldots, l_{n,x}, l_{n,y})^T \]
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Bel(x_t, m_t) = \begin{pmatrix}
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\theta \\
l_1 \\
l_2 \\
\vdots \\
l_n
\end{pmatrix}
\]

- Matrix \( K \) (gain) is not sparse: measuring a landmark improves precision of both landmark position and robot position.
EKF-SLAM

**Prediction**

\[
\text{EKF\_SLAM\_Prediction}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, R_t):
\]

2: \[ F_x = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\end{pmatrix} \]

3: \[ \bar{\mu}_t = \mu_{t-1} + F_x^T \begin{pmatrix}
\frac{-v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin (\mu_{t-1,\theta} + \omega_t \Delta t) \\
\frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \cos (\mu_{t-1,\theta} + \omega_t \Delta t) \\
\omega_t \Delta t \\
\end{pmatrix} \]

4: \[ G_t = I + F_x^T \begin{pmatrix}
0 & 0 & -\frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \cos (\mu_{t-1,\theta} + \omega_t \Delta t) \\
0 & 0 & -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin (\mu_{t-1,\theta} + \omega_t \Delta t) \\
0 & 0 & 0 \\
\end{pmatrix} F_x \]

5: \[ \bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + F_x^T R_t^x F_x \]

\[ \underbrace{R_t}_{F_x} \]
EKF-SLAM

Correction 1

EKF_SLAM_Correction

6: \[ Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{pmatrix} \]

7: for all observed features \( z_t^i = (r_t^i, \phi_t^i)^T \) do

8: \[ j = c_t^i \]

9: if landmark \( j \) never seen before

10: \[ \begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix} \]

11: endif

12: \[ \delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \]

13: \[ q = \delta^T \delta \]

14: \[ \hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \]
EKF-SLAM

Correction 2

15: \[ F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \end{pmatrix}_{2j-2,2N-2j} \]

16: \[ H^i_t = \frac{1}{q} \begin{pmatrix} -\sqrt{q} \delta x & -\sqrt{q} \delta y & 0 & +\sqrt{q} \delta x & \sqrt{q} \delta y \\ \delta y & -\delta x & -q & -\delta y & +\delta x \end{pmatrix} F_{x,j} \]

17: \[ K^i_t = \tilde{\Sigma}_t H^i_t (H^i_t \tilde{\Sigma}_t H^i_t + Q_t)^{-1} \]

18: \[ \bar{\mu}_t = \bar{\mu}_t + K^i_t (z^i_t - \bar{z}^i_t) \]

19: \[ \tilde{\Sigma}_t = (I - K^i_t H^i_t) \tilde{\Sigma}_t \]

20: endfor

21: \[ \mu_t = \bar{\mu}_t \]

22: \[ \Sigma_t = \tilde{\Sigma}_t \]

23: return \( \mu_t, \Sigma_t \)
EKF-SLAM
EKF-SLAM
When correspondences are unknown: Data Association

Was the observation generated by the red or the blue landmark?

$p(\text{observation} | \text{red}) = 0.3$

$p(\text{observation} | \text{blue}) = 0.7$

• Two options for data association:
  • Pick the most probable match.
  • Pick a random association weighted by the observation likelihoods.
  • If the probability is too low, generate a new landmark.

• Each measurement is processed separately: one landmark can correspond to several measurements.
When correspondences are unknown: Data Association

Was the observation generated by the red or the blue landmark?

\[ p(\text{observation}|\text{red} = 0.3) \quad p(\text{observation}|\text{blue} = 0.7) \]

- Two options for data association:
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  - Pick a random association weighted by the observation likelihoods.

- If the probability is too low, generate a new landmark.
- Each measurement is processed separately: one landmark can correspond to several measurements.
EKF-SLAM

- EKF-SLAM is online: previous positions are "hidden" in the covariance matrix.
- Initialization of landmarks:
  - To zero: \((0, 0, 0)\)
  - To the first measurements of the landmark:
    \[
    \begin{pmatrix}
    \mu_{j,x} \\
    \mu_{j,y}
    \end{pmatrix} = \begin{pmatrix}
    \mu_{t,x} \\
    \mu_{t,y}
    \end{pmatrix} + \begin{pmatrix}
    r_t^i \cos(\phi_t^i + \mu_{j,\phi}) \\
    r_t^i \sin(\phi_t^i + \mu_{j,\phi})
    \end{pmatrix}
    \]
  - Bearing only sensors (cameras): integration of several measurements.
- Filtration of outliers: provisional landmark list.
- Landmark existence probability – log odds ratio.
- Numerical instability – initialization of new landmark estimate.
EKF-SLAM: example
Localization vs. SLAM

- A particle filter can be used to solve both problems.
- Localization: state space \( \langle x, y, \theta \rangle \).
- SLAM: state space \( \langle x, y, \theta, \text{map} \rangle \).
  - for landmark maps: \( \langle l_1, l_2, \ldots, l_n \rangle \)
  - for grid maps: \( \langle c_{1,1}, c_{1,2}, \ldots, c_{1,n}, c_{2,1}, \ldots, c_{n,m} \rangle \)
Localization vs. SLAM

- A particle filter can be used to solve both problems.
- Localization: state space $\langle x, y, \theta \rangle$.
- SLAM: state space $\langle x, y, \theta, \text{map} \rangle$.
  - for landmark maps: $\langle l_1, l_2, \ldots, l_n \rangle$
  - for grid maps: $\langle c_{1,1}, c_{1,2}, \ldots, c_{1,n}, c_{2,1}, \ldots, c_{n,m} \rangle$
- **Problem:** The number of particles needed to represent a posterior grows exponentially with the dimension of the state space!
Dependencies
How to reduce dimension of the state space

- Is there a dependency between the dimensions of the state space?
- If so, can we use the dependency to solve the problem more efficiently?
Dependencies

How to reduce dimension of the state space

• Is there a dependency between the dimensions of the state space?
• If so, can we use the dependency to solve the problem more efficiently?
• In the SLAM context
  • The map depends on the poses of the robot.
  • We know how to build a map given the position of the sensor is known.
Factored Posterior Probability (landmarks)

\[ p(x_{1:t}, l_{1:m} | z_{1:t}, u_{0:t-1}) = p(x_{1:t} | z_{1:t}, u_{0:t-1}) p(l_{1:m} | x_{1:t} z_{1:t}) \]

- Observations
- Movements
- Positions
- Landmarks

Does this help to solve the problem?

Factorization first introduced by Murphy in 1999.
Mapping using landmarks

Knowledge of the robot’s true path renders landmark positions conditionally independent!
Factored Posterior

\[ p(x_{1:t}, l_{1:m} | z_{1:t}, u_{0,t-1}) \]

\[ = p(x_{1:t} | z_{1:t}, u_{0,t-1}) p(l_{1:m} | x_{1:t}, z_{1:t}) \]

\[ = p(x_{1:t} | z_{1:t}, u_{0,t-1}) \prod_{i=1}^{m} p(l_i | x_{1:t}, z_{1:t}) \]

Robot path posterior (localization problem) \times

Conditionally independent landmark positions
Rao-Blackwellization

\[ p(x_{1:t}, l_{1:m} | z_{1:t}, u_{0,t-1}) = p(x_{1:t} | z_{1:t}, u_{0,t-1}) \prod_{i=1}^{m} p(l_i | x_{1:t}, z_{1:t}) \]

- This factorization is also called Rao-Blackwellization
- Given that the second term can be computed efficiently, particle filtering becomes possible!
FastSLAM

- Rao-Blackwellized particle filtering based on landmarks (Montemerlo et al., 2002)
- Each landmark is represented by a 2x2 Extended Kalman Filter (EKF).
- Each particle therefore has to maintain $M$ EKFs.
Multi-Hypothesis Data Association

- Data association is done on a per-particle basis.
- Robot pose error is factored out of data association decisions
Results - Victoria Park

- Trajectory length: 4 km
- Position error $< 5$ m RMS
- 100 particles

Blue = GPS
Yellow = FastSLAM

Dataset courtesy of University of Sydney
Results - Data Association

Comparison of FastSLAM and EKF Given Motion Ambiguity

- **FastSLAM**
- **EKF**

**Y-axis:** Robot RMS Position Error (m)

**X-axis:** Error Added to Rotational Velocity (std.)
Results - Accuracy

Accuracy of FastSLAM vs. the EKF on Simulated Data

RMS Pose Error (meters)

Number of Particles
FastSLAM - conclusion

- FastSLAM is both full SLAM and online SLAM.
- Complexity of $O(nm)$ can be improved to $O(n \log m)$.
- Unknown correspondences: similarly to EKF-SLAM (for each particle separately!).
- Motion model does not correspond with the desired distribution $\Rightarrow$ sampling with respect to sensor model.
- Can process also negative information.
FastSLAM - improved proposal

The proposal adapts to the structure of the environment.
Grid-based SLAM

- Can we solve the SLAM problem if no pre-defined landmarks are available?
- Can we use the ideas of FastSLAM to build grid maps?
- As with landmarks, the map depends on the poses of the robot during data acquisition.
- If the poses are known, grid-based mapping is easy (“mapping with known poses”).
Rao-Blackwellization

\[
p(x_{1:t}, m|z_{1:t}, u_{0:t-1}) = p(x_{1:t}|z_{1:t}, u_{0:t-1})p(m|x_{1:t}, z_{1:t})
\]

- **poses**
- **map**
- **observations**
- **movements**

SLAM posterior  Robot path posterior  Mapping with known poses
Rao-Blackwellization

\[ p(x_{1:t}, m|z_{1:t}, u_{0:t-1}) = \]
\[ p(x_{1:t}|z_{1:t}, u_{0:t-1})p(m|x_{1:t}, z_{1:t}) \]

- Use particle filter for localization.
- Use the pose estimate from the MCL part and apply mapping with known poses.
Grid-based FastSLAM

Algorithm

- for $k = 1$ to $m$
  - Select a sample from the previous generation.
  - Update the sample using motion model.
  - Compute the weight according to sensor model.
  - Update the map for the corresponding particle.
FastSLAM with Improved Odometry

- Scan-matching provides a locally consistent pose correction.
- Pre-correct short odometry sequences using scan-matching and use them as input to FastSLAM.
- Fewer particles are needed, since the error in the input is smaller.

[Haehnel et al., 2003]
Conclusion

• The ideas of FastSLAM can also be applied in the context of grid maps.
• Utilizing accurate sensor observation leads to good proposals and highly efficient filters.
• It is similar to scan-matching on a per-particle base.
• The number of necessary particles and re-sampling steps can seriously be reduced.
• Improved versions of grid-based FastSLAM can handle larger environments than naïve implementations in “real time” since they need one order of magnitude fewer samples.
GraphSLAM

- Use a graph to represent the problem
- Every node in the graph corresponds to a pose of the robot/landmark during mapping
- Every edge between two nodes corresponds to a spatial constraint between them
- **Graph-Based SLAM**: Build the graph and find a node configuration that minimizes the error introduced by the constraints
GraphSLAM

Find a configuration of the nodes so that the real and predicted observations are as similar as possible

- Nodes can represent:
  - Robot poses $x_t$
  - Landmark locations $m_j$

- Edges can represent:
  - Landmark observations $z^i_t$
  - Odometry measurements $u_t$

- The minimization optimizes the landmark locations and robot poses
GraphSLAM

Building up the graph

- Information about measurements/controls is mapped into constraints between nodes.
- Edge \( \approx \) spring in a spring-mass model
- Measurement:

\[
(z_t^i - h(x_t, m_j))^T Q_t^{-1} (z_t^i - h(x_t, m_j))
\]

- Control (robot movement)

\[
(x_t - g(x_{t-1}, u_t))^T R_t^{-1} (x_t - g(x_{t-1}, u_t))
\]

\[
J_G = x_0^T \Omega_0 x_0 + \sum_t (x_t - g(x_{t-1}, u_t))^T R_t^{-1} (x_t - g(x_{t-1}, u_t))
\]

\[
+ \sum_t \sum_i (z_t^i - h(x_t, m_j))^T Q_t^{-1} (z_t^i - h(x_t, m_j))
\]
GraphSLAM

Example

\[
J_G = x_0^T \Omega_0 x_0 + \sum_t (x_t - g(x_{t-1}, u_t))^T R_t^{-1} (x_t - g(x_{t-1}, u_t)) \\
+ \sum_t \sum_i (z_t^i - h(x_t, m_j))^T Q_t^{-1} (z_t^i - h(x_t, m_j))
\]
Canonical parametrization of a Gaussian

\[ p(x) = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \]
Canonical parametrization of a Gaussian

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\[ = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu - \frac{1}{2} \mu^T \Sigma^{-1} \mu \right\} \]
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\[ = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \mu^T \Sigma^{-1} \mu \right\} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu \right\} \]

\[ \text{const.} \]
Canonical parametrization of a Gaussian

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\[ = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \mu^T \Sigma^{-1} \mu \right\} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu \right\} \]

\[ const. \]

\[ = \eta \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu \right\} \]
Canonical parametrization of a Gaussian

\[ p(x) = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \]

\[ = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu - \frac{1}{2} \mu^T \Sigma^{-1} \mu \right\} \]

\[ = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \mu^T \Sigma^{-1} \mu \right\} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu \right\} \]

\[ = \eta \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu \right\} \]

\[ = \eta \exp \left\{ -\frac{1}{2} x^T \Omega x + x^T \xi \right\} \]

- Information matrix: \( \Omega = \Sigma^{-1} \)
- Information vector: \( \xi = \Sigma^{-1} \mu \)
GraphSLAM

Information matrix is sparse!

Observation of landmark $m_1$. 
GraphSLAM

Information matrix is sparse!

Robot moves from $x_1$ to $x_2$. 
GraphSLAM

Information matrix is sparse!

everal steps later ...
GraphSLAM
Solving the problem

- Goal: minimize $J_G$
- General (non-linear) least squares form which can be solved by a number of algorithms:
  - Gradient descent
  - Levenberg-Marquardt
  - Conjugate gradient
- g2o (http://openslam.org/g2o.html): A General Framework for Graph Optimization
- These techniques compute mode only (not covariance).
I was inspired by lessons of Sebastian Thrun, from which majority of the presented figures comes. These (and many others) can be found and download from

http://www.probabilistic-robotics.org/.

I also recommend the book


and slides from SLAM Tutorial@ICRA 2016

www.dis.uniroma1.it/~labrococo/tutorial_icra_2016/