Probabilistic decisions

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Notes

(Re-)introduction uncertainty/probability

- Markov Decision Processes (MDP)/RL – uncertainty about outcome of actions
  - *Sequential* decisions (robot/agent goes from s_0 to s_G)
  - $\pi : \mathcal{S} \rightarrow \mathcal{A}$
  - *Policy (Strategy):* knowing what to do for all possible states.
- Now: uncertainty associated with states
  - Different states may have different prior probabilities.
  - The states $s \in \mathcal{S}$ are not directly observable.
  - They need to be inferred from features $x \in \mathcal{X}$.
  - *Single (repeated)* decision $\delta : \mathcal{X} \rightarrow \mathcal{S}$ ($\delta : \mathcal{X} \rightarrow \mathcal{D}$);
  - *Strategy:* knowing how to decide for all possible measurements.
- Decision example, crossing street:
  - x = camera image; $\mathcal{X}$ is the space of all possible images
  - $\mathcal{S}$ = {car, bus, bicycle, truck} approaching
  - I decide to: $\mathcal{D}$ = {go, wait}

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Just a reminder: MDPs, value iteration and policy iteration methods. In RL: temporal difference learning. Now, strictly speaking, we are interested in single decision. Due to its stochastic nature, we understand that anything can happen and we are seeking optimality in a statistical sense - what is the outcome of the decision when repeated.

$\mathcal{S}$ and $\mathcal{D}$ are often the same as we will see later. Yet, it is convenient to keep it separate. States represent the ground-truth, and Decision is the output of the algorithm.

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Decision example: Insure or not? (from late 1980s) [5]

Known about HIV testing: HIV test falsely positive only in 1 case out of 1000.
A doctor calls: "Your HIV test is positive, 999/1000 you will die in 10 years. I'm sorry ...".
Insurance company does not want to insure a married couple.

- Was the doctor right?
- Was the insurance company rational?

$\mathcal{S} = \{\text{healthy}, \text{infected}\}$, $\mathcal{X} = \{\text{positive\_test}, \text{negative\_test}\}$

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$\mathcal{S} = \{\text{healthy}, \text{infected}\}$, $\mathcal{X} = \{\text{positive\_test}, \text{negative\_test}\}$
What is the probability the man is infected?

A: $\frac{1}{1000}$
B: $\frac{999}{1000}$
C: Don't know yet, more info needed, but less than $\frac{1}{2}$
D: Don't know yet, more info needed, but more than $\frac{1}{2}$

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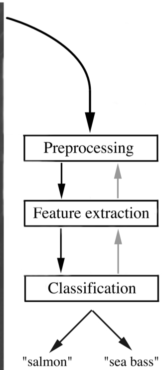
Classification example: What's the fish?



- Factory for fish processing
- 2 classes $s_{1,2}$:
  - salmon
  - sea bass
- Features $\vec{x}$: length, width, lightness etc. from a camera

Notes

- Sea (European) bass, https://en.wikipedia.org/wiki/European_bass. (In Czech it is Mořčák evropský or Mořský vlk.)
- Salmon, https://en.wikipedia.org/wiki/Salmon. (losos in Czech)

Fish – classification using probability

$$posterior = \frac{likelihood \times prior}{evidence}$$

- Notation for classification problem
  - Classes $s_j \in \mathcal{S}$ (e.g., salmon, sea bass)
  - Features $x_i \in \mathcal{X}$ or feature vectors ($\vec{x_i}$) (also called attributes)
- Optimal classification of $\vec{x}$:

$$\delta^*(\vec{x}) = \arg\max_j P(s_j|\vec{x})$$

- We thus choose the most probable class for a given feature vector .
- Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j|\vec{x}) = \frac{P(\vec{x}|s_j)P(s_j)}{P(\vec{x})}$$

- Can we do (classify) better?

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Assuming we know the true $P(\vec{x}|s_j), P(s_j), P(\vec{x})$ we *cannot* do better! Bayesian classification is optimal!

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Decision making under uncertainty

- An important feature of intelligent systems
  - make the best possible decision
  - in uncertain conditions
- **Example:** Take a tram OR subway from A to B?
  - Tram: timetables imply a quicker route, but adherence uncertain.
  - Subway: longer route, but adherence almost certain.
- **Example:** where to route a letter with this ZIP?
  - 15700? 15706? 15200? 15206?
- What is the optimal decision ?
- What is the cost of the decision? What is the associated loss ?
- What is the relation between loss and utility ?

Notes

There are *costs* associated with a decision. E.g. at fish packing plant, customers may not mind so much if some pieces of salmon end up in sea bass cans, but they will be protesting if the opposite happens. So making an error "one way" has higher cost than "the other way". This impacts where decision boundaries for classification should optimally be drawn.

The decision loss can be seen as counterpart of the utility . We want either maximize utitity or minimize loss. In machine learing and pattern recoginition community, the term loss is used much more frequently.

Remember that quality of a policy is evaluated as the expected sum of (discounted) rewards. Quality of the decision strategt will be evaluated based on the expected loss - we will call it risk of the strategy.

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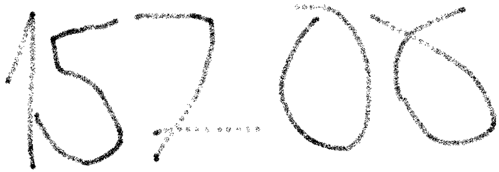

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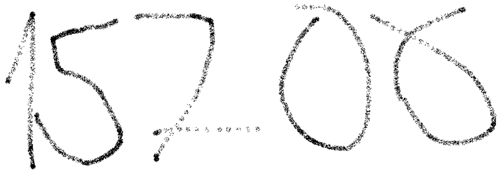

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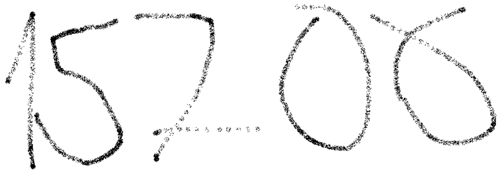

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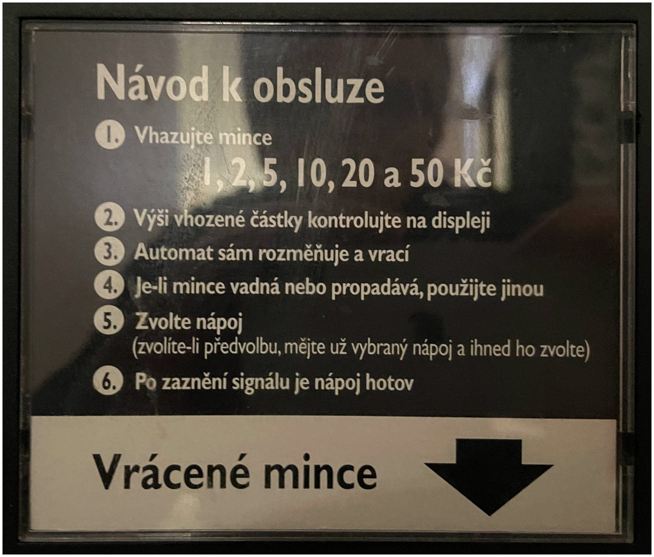
Introducing decision loss: Coin recognition



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Recognizing/classifying coins: components

- $s \in \{1, 2, 5, 10, 20, 50\}$ – state - the true value
- $x \in \{0.0, 0.1, \cdots, 9.9\}[g]$ – measurement, observation
- $P(s, x)$ joint probability
- $d \in \{1, 2, 5, 10, 20, 50\}$– decision, result of the algorithm

Loss function $\ell(?)$ is a function of:

A s

B s, d

C s, x, d

D d

How many strategies?:

A 100

B 100^6

C 600

D 6^{100}

Strategy $d = \delta(?)$ is a function of:

A x

B s

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$P(s, x)$ think about an Oracle for the moment, we will discuss it more later

We assume 100 possible measurements $x \in \{0.0, 0.1, \cdots, 9.9\}$

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Introducing decision loss: What to cook for dinner [4]

- *Wife is coming back from work. Husband: what to cook for dinner?*
- 3 dishes (decisions) in his repertoire:
  - *nothing* ... **don't bother cooking** $\Rightarrow$ no work but makes wife upset
  - *pizza* ... **microwave a frozen pizza** $\Rightarrow$ not much work but won't impress
  - *g.T.c.* ... **general Tso's chicken** $\Rightarrow$ will make her day, but very laborious
- "Hassle" incurred by the individual options depends on wife's mood.
- For each of the 9 possible situations (3 possible decisions $\times$ 3 possible states), the cost is quantified by a loss function $\ell(d, s)$:

$\ell(s, d)$	$d = nothing$	$d = pizza$	$d = g.T.c.$
$s = good$	0	2	4
$s = average$	5	3	5
$s = bad$	10	9	6

The wife's state of mind is an uncertain state.

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Was the state known, the decision would be simple.

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Example (cont'd), State uncertain, symptoms, ...

- Husband's experiment. He tells her he accidentally overtaped their wedding video and observes her reaction.
- Anticipates 4 possible reactions:
  - *mild* ... all right, we keep our memories.
  - *irritated* ... how many times do I have to tell you....
  - *upset* ... Why did I marry this guy?
  - *alarming* ... silence
- The reaction is a measurable attribute/symptom ("feature") of the mind state.
- From experience, the husband knows how probable individual reactions are in each state of mind; this is captured by the joint distribution $P(x, s)$.

$P(x, s)$	$x =$ *mild*	$x =$*irritated*	$x =$ *upset*	$x =$ *alarming*
$s =$ *good*	0.35	0.28	0.07	0.00
$s =$ *average*	0.04	0.10	0.04	0.02
$s =$ *bad*	0.00	0.02	0.05	0.03

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Joint distibution. Husband tried similar experiment multiple times, gathered some evidence ...

Instead of complicated experiment with overtaping the wedding video, think about asking "when are you coming home?".

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  - *irritated* ... how many times do I have to tell you....
  - *upset* ... Why did I marry this guy?
  - *alarming* ... silence
- The reaction is a measurable attribute/symptom ("feature") of the mind state.
- From experience, the husband knows how probable individual reactions are in each state of mind; this is captured by the joint distribution $P(x, s)$.

$P(x, s)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$s = good$	0.35	0.28	0.07	0.00
$s = average$	0.04	0.10	0.04	0.02
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Notes

Joint distibution. Husband tried similar experiment multiple times, gathered some evidence ...

Instead of complicated experiment with overtaping the wedding video, think about asking "when are you coming home?".

Example (cont'd), State uncertain, symptoms, ...

- Husband's experiment. He tells her he accidentally overtaped their wedding video and observes her reaction.
- Anticipates 4 possible reactions:
  - *mild* ... all right, we keep our memories.
  - *irritated* ... how many times do I have to tell you....
  - *upset* ... Why did I marry this guy?
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Decision strategy

- **Decision strategy** : a rule selecting a decision for *any given value* of the measured attribute(s).
- i.e. function $d = \delta(x)$.
- Example of husband's possible strategies:

$\delta(x)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$\delta_1(x) =$	*nothing*	*nothing*	*pizza*	*g.T.c.*
$\delta_2(x) =$	*nothing*	*pizza*	*g.T.c.*	*g.T.c.*
$\delta_3(x) =$	*g.T.c.*	*g.T.c.*	*g.T.c.*	*g.T.c.*
$\delta_4(x) =$	*nothing*	*nothing*	*nothing*	*nothing*

- How many strategies?
- How to define which strategy is the best? How to sort them by quality?
- Define the risk of a strategy as a mean (expected) loss value .

$$r(\delta) = \sum_x \sum_s \ell(s, \delta(x)) P(x, s)$$

Notes

Overall, $3^4 = 81$ possible strategies (3 possible decisions for each of the 4 possible attribute values). There is some analogy of states and possible actions. Here, we reason about states - which are 3 (state of mind) - from features which are 4.
Any given value (of measured attribute) ... Think about any possible state.
Recall MDPs and RL.

- Reward (or penalty) was associated with state or state transition when executing an action $R(s, a, s')$. Similarly here, loss, $\ell(s, \delta(x))$, is associated with state and decision/action.
- Difference: policy / decision strategy.
  - MDP/RL: policy $\pi(s)$
  - Now: state s not directly observable anymore. Instead, policy / decision strategy, $\delta(x)$, needs to be defined over ther *percepts/symptoms/attributes*, x.
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$\ell(s,d)$	$d = nothing$	$d = pizza$	$d = g.T.c.$
$s = good$	0	2	4
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Do we need to evaluate all possible strategies? $P(x,s) = P(s|x)P(x)$

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Calculate $r(\delta_1)$ and $r(\delta_2)$, which strategy is better?

Bayes optimal strategy

- The Bayes optimal strategy : one minimizing mean risk.

$$\delta^* = \arg\min_{\delta} r(\delta)$$

- From $P(x,s) = P(s|x)P(x)$ (Bayes rule), we have

$$r(\delta) = \sum_x \sum_s \ell(s,\delta(x))P(x,s) = \sum_s \sum_x \ell(s,\delta(x))P(s|x)P(x)$$

$$= \sum_x P(x) \underbrace{\sum_s \ell(s,\delta(x))P(s|x)}_{\text{Conditional risk}}$$

- The optimal strategy is obtained by minimizing the conditional risk *separately* for each x:

$$\delta^*(x) = \arg\min_{d} \sum_s \ell(s,d)P(s|x)$$

Notes

Optimal strategy: $\delta^*(x) = \arg\min_d \sum_s \ell(s,d)P(s|x)$

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$\delta(x)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$\delta^*(x) =$	??	??	??	??

Notes

We need to recompute the table of joint probability $P(s,x)$ into table of conditional probabilies $P(s|x)$.
This can be done in two ways. A: Using product rule, $P(s|x) = P(s,x)/P(x)$.

First, to get $P(x)$, we use Sum rule (marginalizing).

	$x = mild$	$x =irritated$	$x = upset$	$x = alarming$
$P(x)$	0.39	0.40	0.16	0.05

Second, applying product rule, $P(s|x) = P(s,x)/P(x)$.
B: calculating the probability on a "per column basis".
E.g. for the first cell, A: $0.35/0.39 = 0.897$ B: $0.35/(0.35 + 0.04)$

$P(s|x)$	$x = mild$	$x =irritated$	$x = upset$	$x = alarming$
$s = good$	0.897	0.7	0.438	0.00
$s = average$	0.103	0.25	0.25	0.4
$s = bad$	0.00	0.125	0.313	0.6

Having the table of all $P(s|x)$ we just mechanically insert into the equation in the slide title.

Statistical decision making: wrapping up

- **Given**:
  - A set of possible states : $\mathcal{S}$
  - A set of possible decisions : $\mathcal{D}$
  - A loss function $\ell : \mathcal{D} \times \mathcal{S} \to \Re$
  - The range $\mathcal{X}$ of the attribute
  - Distribution $P(x, s)$, $x \in \mathcal{X}, s \in \mathcal{S}$.
- **Define**:
  - Strategy : function $\delta : \mathcal{X} \to \mathcal{D}$
  - Risk of strategy δ : $r(\delta) = \sum_x \sum_s \ell(s, \delta(x)) P(x, s)$
- **Bayes problem**:
  - Goal: find the optimal strategy $\delta^* = \arg\min_\delta r(\delta)$
  - Solution: $\delta^*(x) = \arg\min_d \sum_s \ell(s, d) P(s|x)$ (for each x)

Notes

A special case - Bayesian *classification*

- Bayesian classification is a special case of statistical decision theory:
  - Attribute vector $\vec{x} = (x_1, x_2, \ldots)$: pixels 1, 2, ...
  - **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \ldots 9\}$.**
  - State = actual class, Decision = recognized class
  - Loss function:

$$\ell(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{\ell(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg\min_d [1 - P(d|\vec{x})] = \arg\max_d P(d|\vec{x})$$

Notes

- Classification as opposed to Decision
- Loss function simply counts errors (misclassifications)
- We consider all errors equally painful!
- More examples during the lab ...
- The final result is not that surprising, is it? (Is it good or bad?)

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References I

Further reading: Chapter 13 and 14 of [7] (Chapters 12 and 13 in [8]). Books [2] (for this lecture, read Chapter 1) and [3] are classical textbooks in the field of pattern recognition and machine learning. Interesting insights into how people think and interact with probabilities are presented in [5] (in Czech as [6]).

[1] People vs. Collins.
https://law.justia.com/cases/california/supreme-court/2d/68/319.html.

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Notes

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[4] Zdeněk Kotek, Petr Vysoký, and Zdeněk Zdráhal.
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[5] Leonard Mlodinow.
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Notes

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[7] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 3rd edition, 2010.
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[8] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 4th edition, 2021.
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Notes

Additional material for thinking

Notes

Decision: guilty or not? (people of CA vs Collins, 1968) [5]

- Robbery, LA 1964, fuzzy evidence of the offenders:
  - female, around 65 kg
  - wearing something dark
  - hair of light color, between light and dark blond, in a ponytail
- At the same time, additional evidence close to the crime scene:
  - loud scream, yelling, looking at the this direction ...
  - a woman sitting into a yellow car
  - car starts immediately and passes close to the additional witness
  - a black man with beard and moustache was driving
- No more evidence
- Testimony of both the victim and the witness not unambiguous (didn't recognize suspects)
- Still, the suspects were sentenced to jail.

Notes

Wrong use of independence assumption:

$$\begin{aligned} P(\text{yellow car}) &= 1/10 \\ P(\text{man with moustache}) &= 1/4 \\ P(\text{black man with beard}) &= 1/10 \\ P(\text{woman with pony tail}) &= 1/10 \\ P(\text{woman blond hair}) &= 1/3 \\ P(\text{mix race pair in a car}) &= 1/1000 \end{aligned}$$

and mistakenly confusing probability

$$P(\text{randomly selected pair matches discussed characteristics})$$

giving $P = 1/12000000$. Think about total California population.

with the needed conditional probability: $P(\text{a pair matching characteristics is guilty})$

"The court noted that the correct statistical inference would be the probability that no other couple who could have committed the robbery had the same traits as the defendants given that at least one couple had the identified traits. The court noted, in an appendix to its decision, that using this correct statistical inference, even if the prosecutor's statistics were all correct and independent as he assumed, the probability that the defendants were innocent would be over 40%." https://en.wikipedia.org/wiki/People_v._Collins

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Assume (wrong!) mutual indepedence:

$$P(?) = \frac{1}{12,000,000}$$

What probability?

A Convicted pair not guilty.

B A randomly selected pair matches characteristics.

C Some other.

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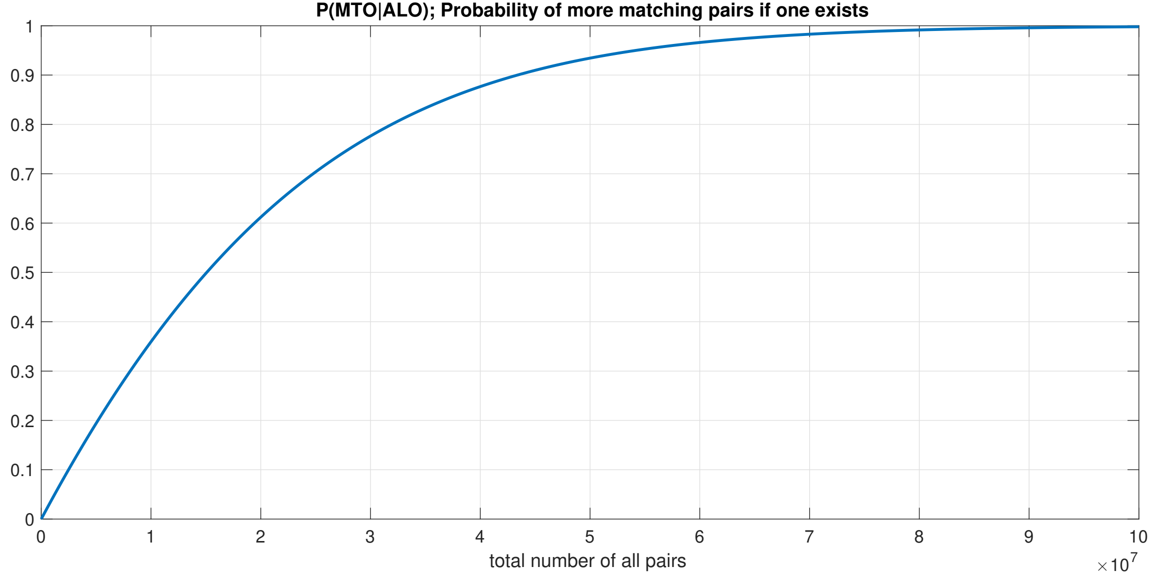
$P(MTO|ALO) = f(N)$; people of CA vs Collins, 1968



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