Reinforcement learning

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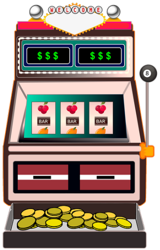
(Multi-armed) Bandits



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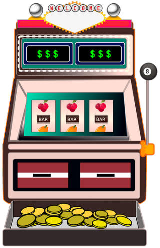
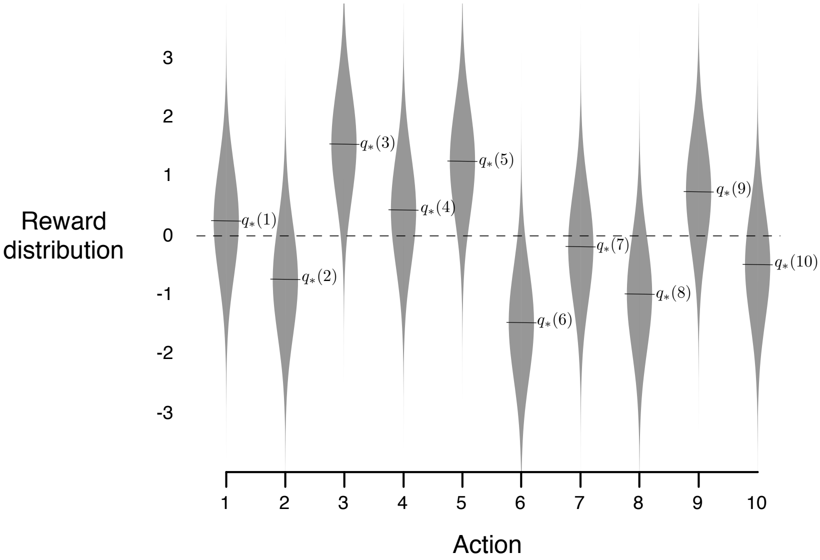
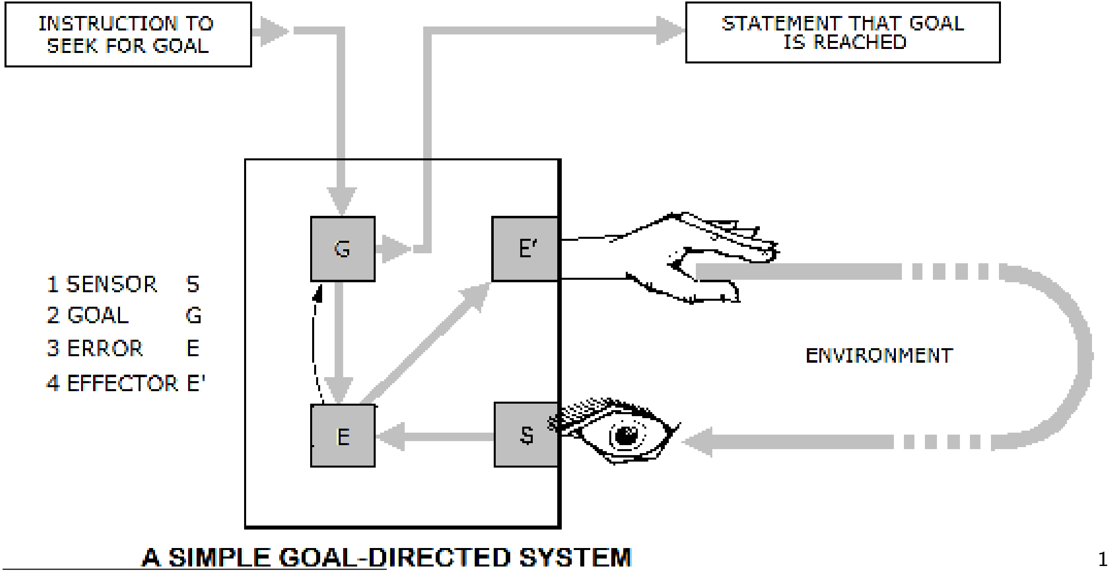


$p(s'|s,a)$ and $r(s,a,s')$ not known!

10 armed bandit, what arm to pull?

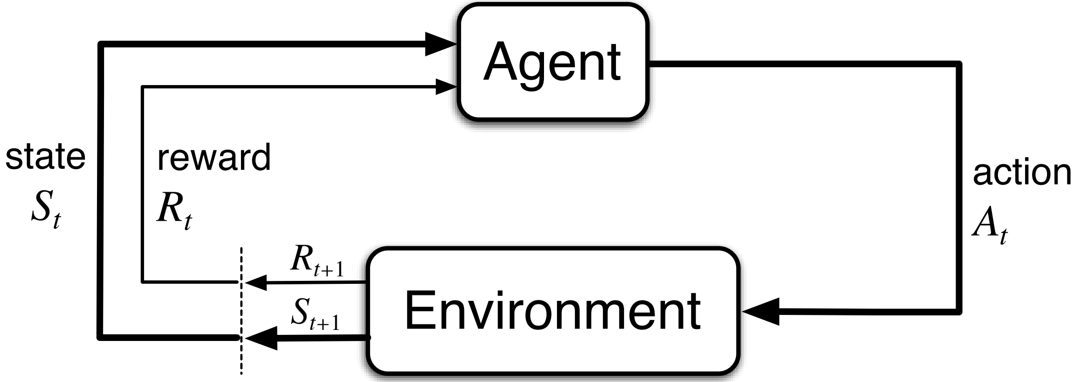


Goal-directed system



[1]Figure from http://www.cybsoc.org/gcyb.htm

Reinforcement Learning



2

- Feedback in form of Rewards
- Learn to act so as to maximize expected rewards.

[2]Scheme from [4]

Examples

Video: Learning safe policies[3]

[3] M. Pecka, V. Salansky, K. Zimmermann, T. Svoboda. Autonomous flipper control with safety constraints. In Intelligent Robots and Systems (IROS), 2016, https://youtu.be/_oUMbBtoRcs

From off-line (MDPs) to on-line (RL)

Markov decision process – MDPs. Off-line search, we know:

- A set of states $s \in \mathcal{S}$ (map)
- A set of actions per state. $a \in \mathcal{A}$
- A transition model $T(s, a, s')$ or $p(s'|s, a)$ (robot)
- A reward function $r(s, a, s')$ (map, robot)

Looking for the optimal policy $\pi(s)$. We can plan/search before the robot enters the environment.

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Looking for the optimal policy $\pi(s)$. We can plan/search before the robot enters the environment.

On-line problem:

- Transition model p and reward function r not known.
- Agent/robot must act and learn from experience.

(Transition) Model-based learning

The main idea: Do something and:

- Learn an approximate model from experiences.
- Solve as if the model was correct.

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Learning MDP model:

- In s try a, observe s', count (s, a, s').
- Normalize to get and estimate of $p(s' \mid s, a)$.
- Discover (by observation) each $r(s, a, s')$ when experienced.

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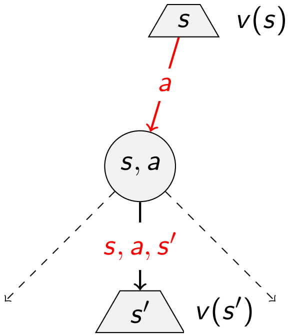
Solve the learned MDP.

Reward function $r(s, a, s')$

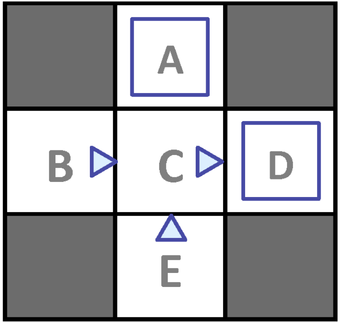
- $r(s, a, s')$ - reward for taking a in s and landing in s'.
- In Grid world, we assumed $r(s, a, s')$ to be the same everywhere.
- In the real world, it is different (going up, down, ...)



In ai-gym `env.step(action)` returns s', $r(s, \texttt{action}, s')$.

Model-based learning: Grid example

Input Policy π



Assume: $\gamma = 1$

Observed Episodes (Training)

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
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Episode 3

E, north, C, -1
C, east, D, -1
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Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

[4]

[4] Figure from [1]

Learning transition model

$p(\mathrm{D} \mid \mathrm{C}, \mathrm{east}) = ?$

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Learning reward function

$r(\mathrm{C}, \mathrm{east}, \mathrm{D}) = ?$

Episode 1

B, east, C, -1
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Model based vs model-free: Expected age $\mathrm{E}[A]$

Random variable age A.

$$\mathrm{E}[A] = \sum_a P(A = a)a$$

We do not know $P(A = a)$. Instead, we collect N samples $[a_1, a_2, \dots a_N]$.

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Model based

$$\hat{P}(a) = \frac{\mathrm{num}(a)}{N}$$

$$\mathrm{E}[A] \approx \sum_a \hat{P}(a)a$$

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Model based

$$\hat{P}(a) = \frac{\text{num}(a)}{N}$$

$$\mathrm{E}[A] \approx \sum_a \hat{P}(a)a$$

Model free

$$\mathrm{E}[A] \approx \frac{1}{N} \sum_i a_i$$

Model-free learning

Passive learning (evaluating given policy)

- **Input:** a fixed policy $\pi(s)$
- We want to know how good it is.
- r, p not known.
- Execute policy ...
- and learn on the way.
- **Goal:** learn the state values $v^{\pi}(s)$

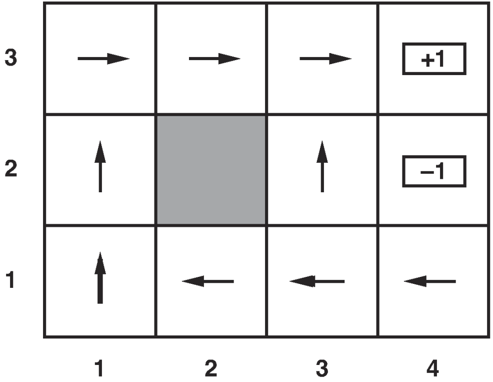


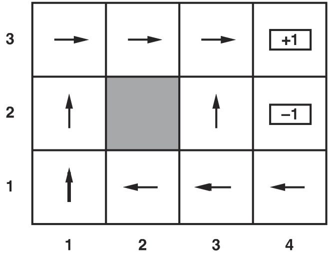
Image from [2]

Direct evaluation from episodes

Value of s for π – expected sum of discounted rewards – expected return

$$v^{\pi}(S_t) = \mathrm{E}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

$$v^{\pi}(S_t) = \mathrm{E}\left[G_t\right]$$



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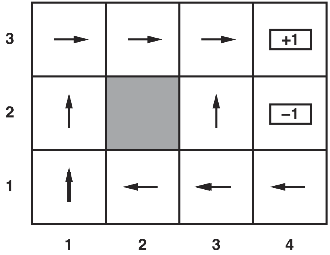

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Direct evaluation from episodes, $v^\pi(S_t) = \mathrm{E}\left[G_t\right]$, $\gamma = 1$

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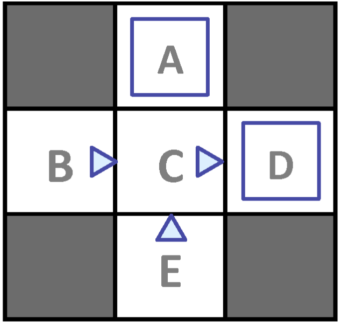
What is $v(3,2)$ after these episodes?

Direct evaluation: Grid example

Input Policy π



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Direct evaluation: Grid example, $\gamma = 1$

What is $v(C)$ after the 4 episodes?

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
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Episode 3

E, north, C, -1
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Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

Direct evaluation: Grid example, $\gamma = 1$

What is $v(\mathrm{C})$ after the 4 episodes?

Let M be the number of recorded episodes.
Let N be the number of samples used to compute the averages.
What is the relation of M and N?

A $N = M$

B $N \leq M$

C $N \geq M$

D N has no relation to M

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

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Direct evaluation algorithm (every-visit version)

$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
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Input: a policy π to be evaluated
Initialize:
    $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$
    $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$
Loop forever (for each episode):
    Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$
    $G \leftarrow 0$
    Loop backwards for each step of episode, $t = T-1, T-2, \ldots, 0$:
        $G \leftarrow R_{t+1} + \gamma G$
        Append G to $Returns(S_t)$
        $V(S_t) \leftarrow \text{average}(Returns(S_t))$

Direct evaluation algorithm (first-visit version)

$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
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  $G \leftarrow 0$
  Loop backwards for each step of episode, $t = T-1, T-2, \ldots, 0$:
    $G \leftarrow R_{t+1} + \gamma G$
    If S_t does not appear in $S_0, S_1, \ldots, S_{t-1}$: *// Use the return for the first visit only*
      Append G to $Returns(S_t)$
      $V(S_t) \leftarrow$ average($Returns(S_t)$)

Direct evaluation: analysis

The good:

- Simple, easy to understand and implement.
- Does not need p, r and eventually it computes the true v^{π}.

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The bad:

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- Each state value learned in isolation.
- State values are not independent
- $v^\pi(s) = \sum_{s'} p\big(s' \mid s, \pi(s)\big)\big[r(s, \pi(s), s') + \gamma\, v^\pi(s')\big]$

(on-line) Policy evaluation?

In MDP, we did:

- Initialize the values: $V_0^\pi(s) = 0$
- In each iteration, replace V with a one-step-look-ahead:
  $V_{k+1}^\pi(s) \leftarrow \sum_{s'} p\big(s' \mid s, \pi(s)\big)\big[r(s, \pi(s), s') + \gamma\, V_k^\pi(s')\big]$

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Problem: both $p\big(s' \mid s, \pi(s)\big)$ and $r(s, \pi(s), s')$ unknown!

Use samples for evaluating policy?

MDP (p, r known) : Update V estimate by a weighted average:
$V^{\pi}_{k+1}(s) \leftarrow \sum_{s'} p\big(s' \mid s, \pi(s)\big)\big[r(s, \pi(s), s') + \gamma\, V^{\pi}_{k}(s')\big]$

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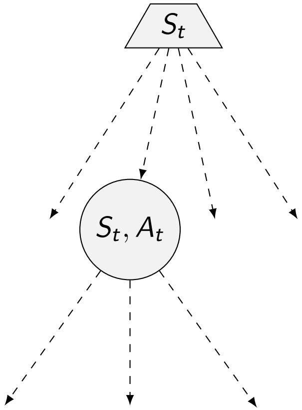
What about stop, try, try, ..., and average?

Trials at time t. $\pi(S_t) \rightarrow A_t$, repeat A_t.



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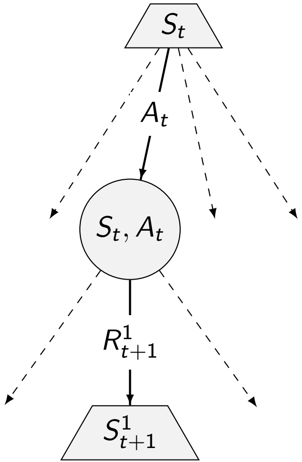
What about stop, try, try, ..., and average?

Trials at time t. $\pi(S_t) \rightarrow A_t$, repeat A_t.

$$\text{trial}^1 \quad = \quad R^1_{t+1} + \gamma\, V(S^1_{t+1})$$



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What about stop, try, try, . . . , and average?
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$$
\begin{aligned}
\text{trial}^1 &= R^1_{t+1} + \gamma\, V(S^1_{t+1}) \\
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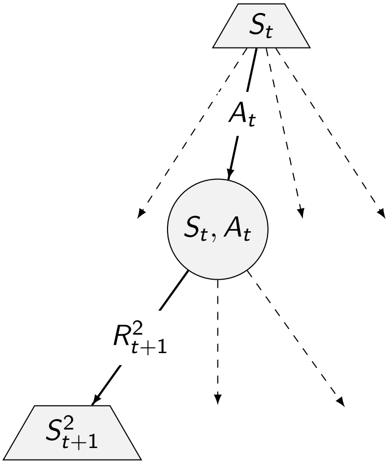
\end{aligned}
$$



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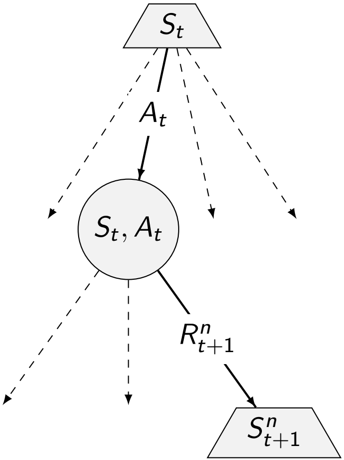
What about stop, try, try, ..., and average?

Trials at time t. $\pi(S_t) \to A_t$, repeat A_t.

$$
\begin{aligned}
\text{trial}^1 &= R^1_{t+1} + \gamma\, V(S^1_{t+1}) \\
\text{trial}^2 &= R^2_{t+1} + \gamma\, V(S^2_{t+1}) \\
\vdots &= \vdots \\
\text{trial}^n &= R^n_{t+1} + \gamma\, V(S^n_{t+1})
\end{aligned}
$$



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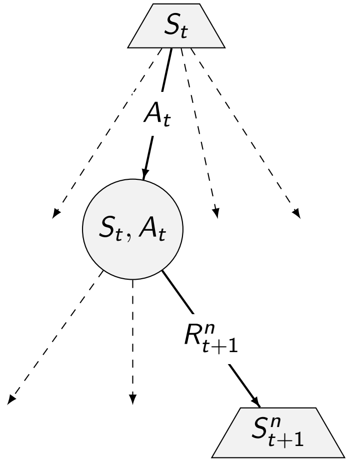
Trials at time t. $\pi(S_t) \rightarrow A_t$, repeat A_t.

$$\begin{aligned}
\text{trial}^1 &= R^1_{t+1} + \gamma\, V(S^1_{t+1}) \\
\text{trial}^2 &= R^2_{t+1} + \gamma\, V(S^2_{t+1}) \\
\vdots &= \vdots \\
\text{trial}^n &= R^n_{t+1} + \gamma\, V(S^n_{t+1})
\end{aligned}$$

$$V(S_t) \leftarrow \frac{1}{n} \sum_i \text{trial}^i$$



Use samples for evaluating policy?

MDP (p, r known) : Update V estimate by a weighted average:
$V^{\pi}_{k+1}(s) \leftarrow \sum_{s'} p\big(s' \mid s, \pi(s)\big)\big[r(s, \pi(s), s') + \gamma\, V^{\pi}_{k}(s')\big]$

What about stop, try, try, ..., and average?
Trials at time t. $\pi(S_t) \rightarrow A_t$, repeat A_t.

$$\begin{aligned}
\text{trial}^1 &= R^1_{t+1} + \gamma\, V(S^1_{t+1}) \\
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\vdots &= \vdots \\
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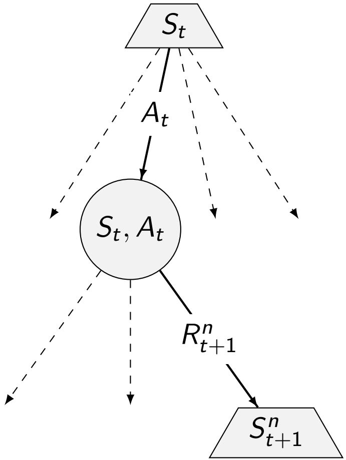

Problem: We cannot re-set to S_t easily.

Temporal-difference value learning

$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (3,2)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
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$\gamma = 1$

Temporal-difference value learning

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$\gamma = 1$
From first trial (episode): $V(2,3) =$, $V(1,3) =$, ...

Temporal-difference value learning

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From first trial (episode): $V(2,3) = 0.92$, $V(1,3) = 0.84, \ldots$

Temporal-difference value learning

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In second episode, going from $S_t = (1,3)$ to $S_{t+1} = (2,3)$ with reward $R_{t+1} = -0.04$, hence:

$$V(1,3) = R_{t+1} + V(2,3) = -0.04 + 0.92 = 0.88$$

Temporal-difference value learning

$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
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Temporal-difference value learning

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- Update ($\alpha\times$ difference): $V(S_t) \leftarrow V(S_t) + \alpha\Big(\big[R_{t+1} + \gamma V(S_{t+1})\big] - V(S_t)\Big)$
- α is the learning rate.

Temporal-difference value learning

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- α is the learning rate.
- $V(S_t) \leftarrow (1-\alpha)V(S_t) + \alpha\,(\text{new sample})$

Exponential moving average

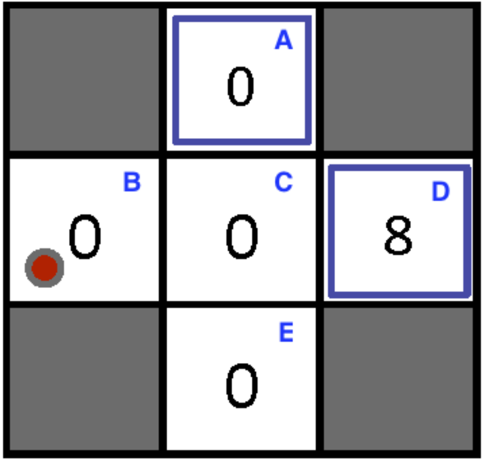
$$\overline{x}_n = (1-\alpha)\overline{x}_{n-1} + \alpha x_n$$

What does it remember about the past? Try to derive:

$$\overline{x}_n = f(\alpha, x_n, x_{n-1}, x_{n-2}, x_{n-3}, \dots)$$

Example: TD Value learning

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$



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- Assume: $\gamma = 1, \alpha = 0.5, \pi(s) = \rightarrow$

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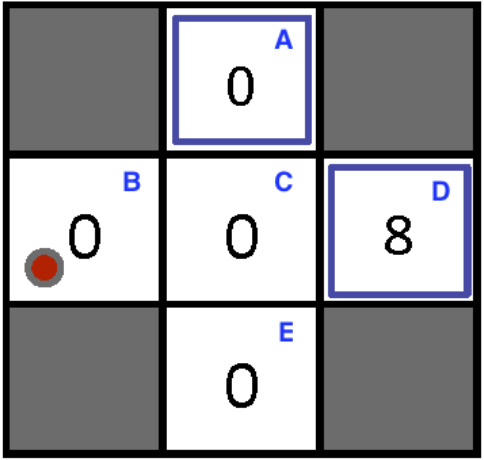

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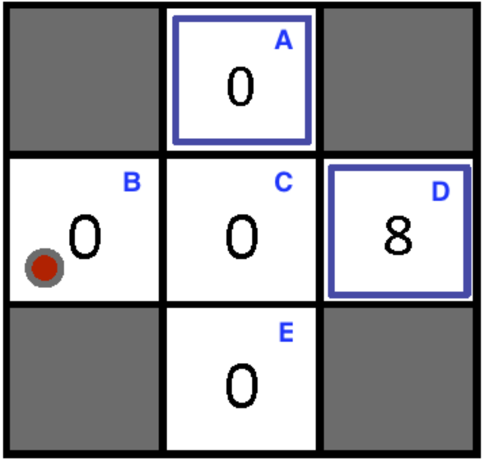

- Values represent initial $V(s)$
- Assume: $\gamma = 1, \alpha = 0.5, \pi(s) = \rightarrow$
- $(B, \rightarrow, C), -2, \Rightarrow V(B)$?
- $(C, \rightarrow, D), -2, \Rightarrow V(C)$?

Temporal difference value learning: algorithm

Input: the policy π to be evaluated
Algorithm parameter: step size $\alpha \in (0, 1]$
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\textit{terminal}) = 0$

Loop for each episode:
    Initialize S
    Loop for each step of episode:
        $A \leftarrow$ action given by π for S
        Take action A, observe R, S'
        $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$
        $S \leftarrow S'$
    until S is terminal

What is wrong with the temporal difference Value learning?

The Good: Model-free value learning by mimicking Bellman updates.

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The Bad: How to turn values into a (new) policy?

- $\pi(s) = \arg\max_a \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V(s') \right]$

What is wrong with the temporal difference Value learning?

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- $\pi(s) = \arg\max_a \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V(s') \right]$
- $\pi(s) = \arg\max_a Q(s, a)$

Active reinforcement learning

Reminder: V, Q-value iteration for MDPs

Value/Utility iteration (depth limited evaluation):

- Start: $V_0(s) = 0$
- In each step update V by looking one step ahead:
  $V_{k+1}(s) \leftarrow \max_a \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V_k(s')\right]$

Q values more useful (think about updating π)

- Start: $Q_0(s, a) = 0$
- In each step update Q by looking one step ahead:
  $Q_{k+1}(s, a) \leftarrow \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma \max_{a'} Q_k(s', a')\right]$

Q-learning

MDP update: $Q_{k+1}(s,a) \leftarrow \sum_{s'} p(s' \mid s,a) \left[r(s,a,s') + \gamma \max_{a'} Q_k(s',a') \right]$

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Learn Q values as the robot/agent goes (temporal difference)

- Drive the robot and fetch rewards (s, a, s', R)

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Learn Q values as the robot/agent goes (temporal difference)

- Drive the robot and fetch rewards (s, a, s', R)
- We know old estimates $Q(s,a)$ (and $Q(s',a')$), if not, initialize.

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Learn Q values as the robot/agent goes (temporal difference)

- Drive the robot and fetch rewards (s, a, s', R)
- We know old estimates $Q(s,a)$ (and $Q(s',a')$), if not, initialize.
- A new trial/sample estimate at time t

  $\text{trial} = R_{t+1} + \gamma \max_a Q(S_{t+1}, a)$

Q-learning

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- α update

  $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(\text{trial} - Q(S_t, A_t))$

  or (the same)

  $Q(S_t, A_t) \leftarrow (1-\alpha)Q(S_t, A_t) + \alpha\,\text{trial}$

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In each step Q approximates the optimal q^* function.

Q-learning: algorithm

step size $0 < \alpha \leq 1$
initialize $Q(s, a)$ for all $s \in \mathcal{S}, a \in \mathcal{S}(s)$
repeat episodes:
    initialize S
    for for each step of episode: **do**
        choose A from S
        take action A, observe R, S'
        $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
        $S \leftarrow S'$
    end for until S is terminal
until Time is up, . . .

From Q-learning to Q-learning agent

- Drive the robot and fetch rewards. (s, a, s', R)
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Technicalities for the Q-learning agent

From Q-learning to Q-learning agent

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Technicalities for the Q-learning agent

- How to represent the Q-function?

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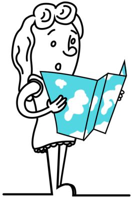
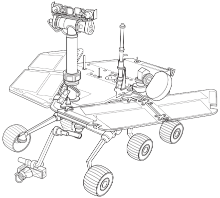
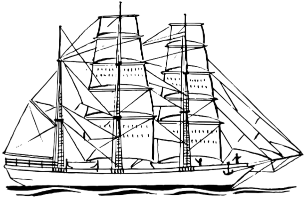
Technicalities for the Q-learning agent

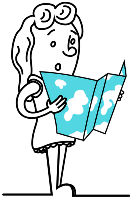
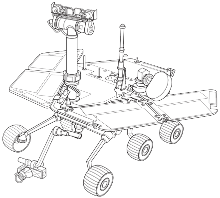
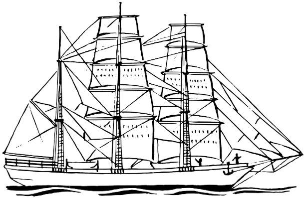
- How to represent the Q-function?
- What is the value for terminal? $Q(s, \text{Exit})$ or $Q(s, \text{None})$
- How to drive? Where to drive next? Does it change over the course?

Exploration vs. Exploitation

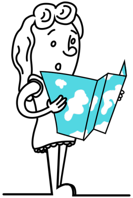
- Drive the known road or try a new one?

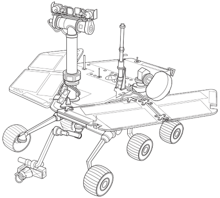
Exploration vs. Exploitation

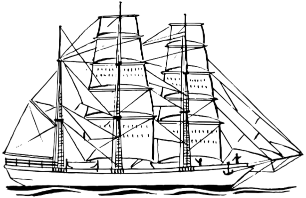
  

- Drive the known road or try a new one?
- Go to the university menza or try a nearby restaurant?

Exploration vs. Exploitation







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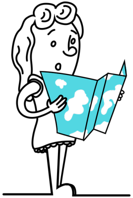
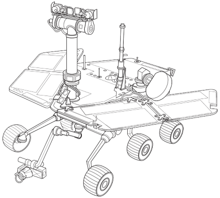
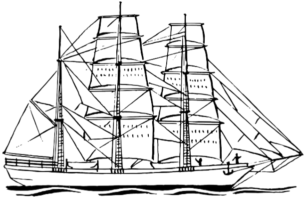
- Use the SW (operating system) I know or try a new one?

Exploration vs. Exploitation







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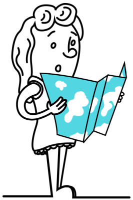
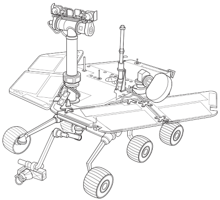
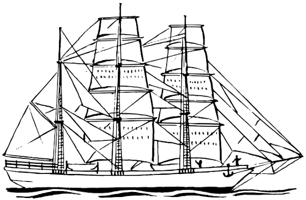

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- ϵ same everywhere?

References I

Further reading: Chapter 21 of [2] (chapter 23 of [3]). More detailed discussion in [4], chapters 5 and 6.

[1] Dan Klein and Pieter Abbeel.
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