Quantum Computing

Exercises 7: Quantum Fourier Transforms

The Quantum Fourier Transform acting on some state $|j\rangle$ is given by

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle.$$

$N = 2^n$ for n qubits. Or, in the tensor-product representation

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} (|0\rangle + e^{2\pi i \frac{j}{2^1}} |1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i \frac{j}{2^n}} |1\rangle)$$

1. *How does the QFT act on $|0\rangle$ and $|1\rangle$? Find the QFT matrix for 1-qubit.*

It acts in the following way

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}} (e^{2\pi i(0)(0)/2} |0\rangle + e^{2\pi i(0)(1)/2} |1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = |+\rangle$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}} (e^{2\pi i(1)(0)/2} |0\rangle + e^{2\pi i(1)(1)/2} |1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = |-\rangle$$

So the matrix can be written as

$$\text{QFT}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which we also call the Hadamard gate.

2. *Show that the Quantum Fourier Transform acting on the n-qubit $|0\rangle^{\otimes n}$ state is equivalent to applying a Hadamard transform to each qubit, which we can write as*

$$H^{\otimes n} |x\rangle = \frac{1}{2^n} \sum_{k=0}^{2^n-1} (-1)^{x \cdot k} |k\rangle.$$

In the integer representation $|0\rangle^{\otimes n}$ is the same as $|0\rangle_n$

$$QFT_n |0\rangle_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i(0)\cdot k/N} |k\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |k\rangle.$$

Which is just the uniform superposition of all possible states.
We know from previous exercises that for a general n-bitstring we have

$$H^{\otimes n} |x\rangle = \frac{1}{2^n} \sum_{k=0}^{2^n-1} (-1)^{x \cdot k} |k\rangle.$$

In our case $x = 0$ and subbing this in we get the same result as above.

3. *Directly proof that the general Quantum Fourier Transform is a unitary transformation.*
Hint: You may need to use the formula for a finite geometric series

$$\sum_{k=0}^{N-1} ar^k = a \left(\frac{1 - r^N}{1 - r} \right)$$

The inverse of the QFT can be expressed as

$$\langle j| QFT^\dagger = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \langle k| e^{-2\pi ijk/N}$$

.
So if QFT is unitary we should have that $(QFT)(QFT)^\dagger = (QFT)^\dagger(QFT) = I$. Hence $\langle j'|(QFT)^\dagger(QFT)|j\rangle = \langle j'|j\rangle = \delta_{jj'}$ since states in this basis are orthonormal.

$$\langle j'|(QFT)^\dagger(QFT)|j\rangle = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{k'=0}^{N-1}\langle k'|e^{2\pi i(jk-j'k')/N}|k\rangle = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{k'=0}^{N-1} e^{2\pi i(jk-j'k')/N}\underbrace{\langle k'|k\rangle}_{\delta_{kk'}}.$$

Hence only non-zero when $k = k'$ so can remove one of the sums

$$= \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i(j-j')k/N}$$

Now consider the first case where $j = j'$. Here the argument of the exponent is 0 and so we are simply just summing 1 N times. In other words

$$\langle j|(QFT)^\dagger(QFT)|j\rangle = \frac{1}{N}\cdot N = 1$$

.

In the case where $j \neq= j'$ we now have a finite geometric series, which is know to always converge as given by the formula above with first term $a = 1$ and common ratio $r = exp(2\pi i(j-j')/N)$. Plugging this into the formula above

$$\langle j'|(QFT)^\dagger(QFT)|j\rangle = \frac{1}{N}\left(\frac{1-(e^{2\pi i(j-j')/N})^N}{1-e^{2\pi i(j-j')/N}}\right) = \frac{1}{N}\left(\frac{1-e^{2\pi i(j-j')}}{1-e^{2\pi i(j-j')/N}}\right)$$

Since $j - j' = z$ is always an integer and looking at the complex unit circle we know that this is just z full rotations around the unit circle i.e. $e^{2\pi iz} = e^{2\pi i}$. Hence the term of the numerator is just 0. All together, we have

$$\langle j|(QFT)^\dagger(QFT)|j\rangle = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j' \end{cases} = \delta_{jj'}$$

4. *Using both representations compute the output of applying the Quantum Fourier Transform on the state* $|5\rangle_3$ *($n = 3$ qubits).*

In the first formula we have

$$QFT_3|5\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{N-1} e^{2\pi i\cdot 5\cdot k/8}|k\rangle = \frac{1}{\sqrt{8}}(|0\rangle+e^{5\pi i/4}|1\rangle+e^{5\pi i/2}|2\rangle+e^{15\pi i/4}|3\rangle+e^{5\pi i}|4\rangle+e^{25\pi i/4}|5\rangle+e^{15\pi i/2}|6\rangle+e^{35\pi i/4}|7\rangle).$$

Using the tensor representation we have

$$FT_3|5\rangle_3 = \frac{1}{\sqrt{8}}(|0\rangle + e^{2\pi i\cdot 5/2}|1\rangle)\otimes(|0\rangle + e^{2\pi i\cdot 5/4}|1\rangle)\otimes(|0\rangle + e^{2\pi i\cdot 5/8}|1\rangle)$$

Simplifying and using the fact that

$$e^{x_1}|x_1\rangle \otimes e^{x_2}|x_2\rangle = e^{x_1+x_2}|x_1x_2\rangle,$$

We get the expression

$$QFT_3|5\rangle_3 = (|000\rangle + e^{5\pi i/4}|001\rangle + e^{5\pi i/2}|010\rangle + e^{15\pi i/4}|011\rangle + e^{5\pi i}|100\rangle + e^{25\pi i/4}|101\rangle + e^{15\pi i/2}|110\rangle + e^{35\pi i/4}|111\rangle)$$

We then just need to convert these binary values to decimal integers. For a binary string $x = x_1x_2\dots x_n$, the decimal representation is $\sum_{k=1}^{n} x_k 2^{n-k}$. Following this through we find that we get the same output as the first representation.