Quantum Computing 2026 - Exercise Sheet 1

Basics of Quantum Mechanics

1. *Given the orthonormal basis states* $\left\{ |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

a) Show that the 'in' and 'out' states defined as:

$$|i\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)$$

$$|o\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$$

are orthonormal.

b) Show that the 'left' and 'right' states defined as:

$$|l\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$$

$$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$$

are orthonormal.

c) Show that the expressions for calculating the expectation values of an operator, $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_n a_n P(|a_n\rangle)$ *where* $a_n, |a_n\rangle$ *are the eigenvalues and eigenvectors respectively.*

d) Calculate the expectation values of σ_y *in the states* $|u\rangle$ *and* $|i\rangle$*, and of* σ_z *in the state* $|o\rangle$, $|l\rangle$.

2. *a) Normalise the state*

$$|\psi\rangle = 3i|u\rangle + (1 - 2i)|d\rangle.$$

b) For this (normalised) state, calculate the probability of getting both positive $(+1)$ *and negative* (-1) *spin eigenvalues by measuring* σ_z.

3. *(Operators and Measurements) Find the eigenvalues, eigenvectors and diagonal representations for the following operators:*

a) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *b)* $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ *c)* $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(d) *Write the above in matrices in terms of Dirac notation: i) in the basis* $\{|u\rangle, |d\rangle\}$ *and ii) in the basis of their eigenvectors.*

(e) *For the wavefunction* $|\psi\rangle = \sqrt{\frac{2}{3}}|u\rangle + \sqrt{\frac{1}{3}}|d\rangle$*, calculate the expectation value for* σ_y

(f) *Show that for each of the sets of orthonormal (show this if you like) eigenvectors the completeness relation*

$$\mathbb{I}_n = \sum_i |\psi_i\rangle\langle\psi_i|$$

is satisfied.

(g) *A quantum state can also be written as* $|\psi\rangle = \cos(\frac{\theta}{2})\,|u\rangle + e^{i\varphi}\sin(\frac{\theta}{2})\,|d\rangle$*, where the angles can be seen in the Bloch sphere*

Draw each of the eigenvectors and the state $|\psi\rangle = \sqrt{\frac{3}{4}}|u\rangle - i\sqrt{\frac{1}{4}}|d\rangle$ *on the Bloch sphere.*

4. *(Properties of Hermitian and Unitary Matrices) For a hermitian matrix* $\mathbf{A}$*, that is, a matrix that satisfies* $\mathbf{A} = \mathbf{A}^\dagger$*, show that:*

(a) *Different eigenvalues have orthogonal eigenvectors.*

(b) *All its eigenvalues are real. Does the converse also hold, that is, if the spectrum (the set of all eigenvalues) of a matrix is in* $\mathbb{R}$*, is it then a hermitian matrix?*

Now, consider a unitary matrix, one for which

$$UU^\dagger = \mathbb{I} \iff U^\dagger U = \mathbb{I} \iff U^{-1} = U^\dagger$$

(c) *Prove that its eigenvalues are of the form* $e^{i\theta}$

(d) *Prove that eigenvectors of different eigenvalues must be orthogonal*

(e) *Now consider two operators* $\mathbf{A}, \mathbf{B}$. *Show that if they are simultaneously diagonalizable, then* $[\mathbf{A}, \mathbf{B}] = 0$

(f) *For any observables* $\mathbf{A}$ *and* $\mathbf{B}$, *and state* $|\psi\rangle$, *derive Heisenberg's uncertainty relation:* $\Delta\mathbf{A}\cdot\Delta\mathbf{B} \geq \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|$, *where* $(\Delta\mathbf{A})^2 = \sum_a (a - \langle\mathbf{A}\rangle)^2 P(a)$, *is the standard deviation of the operator* $\mathbf{A}$. *To do this, first show that:*

  (I) $(\Delta\mathbf{A})^2 = \langle\bar{\mathbf{A}}^2\rangle$ *where* $\bar{\mathbf{A}} = \mathbf{A} - \langle\mathbf{A}\rangle$

  (II) $[\mathbf{A}, \mathbf{B}] = [\bar{\mathbf{A}}, \bar{\mathbf{B}}]$

  (III) *Now, using these and the Cauchy Schwartz inequality,* $2|X||Y| \geq |\langle X|Y\rangle + \langle Y|X\rangle|$, *and defining the states* $|X\rangle = \bar{\mathbf{A}}|\Psi\rangle$, $|Y\rangle = i\bar{\mathbf{B}}|\Psi\rangle$, *derive the uncertainty principle.*

(g) *Calculate the commutation relations between different combinations of the Pauli-matrices* $\sigma_x, \sigma_y, \sigma_z$. *(Bonus: Can you derive some formula for all possible combination?)*

(h) *Can* σ_x *and* σ_y *be simultaneously observed?*

(i) *Consider a one-dimensional quantum particle in the position representation. The position operator acts as*

$$\hat{x}\psi(x) = x\,\psi(x),$$

and the momentum operator acts as

$$\hat{p}\psi(x) = -i\hbar\frac{d}{dx}\psi(x).$$

Compute the commutator $[\hat{x}, \hat{p}]$.