

CHAPTER 3

Quantum Physics

Quantum mechanics was developed in the beginning of the last century as a means to explain certain mystical phenomena observed in experiments involving atoms and light. This led to a revolution in physics and, more broadly, in how we look upon the nature of reality.

Nowadays, even though quantum physics still might sound mysterious and abstract to the uninitiated, it is very much a vital part of our daily lives through its many applications in modern technologies.

Classical physics¹ is completely deterministic. It is possible to know everything about a classical system, and furthermore, once we know enough about the system, we can determine everything about its future through the basic laws of classical physics.

One of the mysterious, or some would even say disturbing, facts about quantum mechanics is that this is no longer true. Quantum mechanics is inherently a non-deterministic, or probabilistic, theory.

In this Chapter we will give a lightning introduction to the wonderful world of quantum mechanics, with of course a special eye towards the applications into computer science. The mathematical language of quantum mechanics is mainly that of linear algebra, so much of the material will in some sense be a review of concepts from linear algebra, but perhaps with a slightly different notation and language than you are familiar with.

We will also discuss the probabilistic nature of quantum mechanics and how this affects results of measurements; how quantum systems evolve with time; the quantum harmonic oscillator; and finally, we will discuss the quantum analogue of the classical bit of computer science.

1. Quantum states

1.1. States, probability and measurements in a classical world. In classical physics, a state takes values in a set. We can for example imagine throwing a die, or flipping a coin. The resulting state will take values either in $\{1, 2, 3, 4, 5, 6\}$ or $\{\text{Heads}, \text{Tails}\}$, respectively. Obviously, it does not make sense to say that the coin is in a mixture of heads and tails.² It simply is in either the state heads or the state tails.

Furthermore, making a measurement in classical physics simply amount to looking at the die or coin to see if it landed on a six, or on heads. Doing this does not change the system in any way. The dice is in the state six before we look at it, and it continues to be so after we have looked at it.

The notion of probability is of course something we occasionally use to describe systems in the classical world as well. After all, playing board games would perhaps be a bit less fun if we always knew exactly how the dice would land. However, this notion of probability is simply a measure of how little information we have about the system. If we had some super computer that could completely characterize the initial state of the dice in the throwers hand, the force and angles of the hand that throws the dice, the wind speed in the room the air is thrown, and so on, it could determine exactly how the dice would land. In classical physics, knowing everything about a system really means knowing everything. Using the laws of classical physics (and given a powerful enough computer) we can completely determine the future of any system once we know enough data.

¹In this course, when we talk about classical physics we simply mean *not* quantum physics.

²If we are very unlucky, it could of course happen that the coin manages to balance on its edge in the end, but then we would simply enlarge the set of values it can take to account for this.

All of this hopefully seems rather obvious to you and you might wonder why we are discussing such basic facts. Well, as we will see, when we step in to the quantum world, these basic things will no longer hold true.

1.2. Quantum states. One of the main differences between classical and quantum physics is the fact that quantum states are not just elements of a set, they are vectors in a complex valued vector space. The strange thing is that we can give some meaning to the statement that a quantum state is in a mixture of states. If we had a quantum coin it could of course be either in the states heads or tails, but it could also be in a mixture of the two. This is called *superposition*.

To see how this works, we first introduce some notation. We imagine that we have a system that is in some state, which we simply label by the letter ψ . This could for example be a number corresponding to one of the classical states $\{1, 2, 3, 4, 5, 6\}$ of a die, but it could also be something else, like \uparrow or \downarrow . The state vector is then denoted as

$$|\psi\rangle.$$

This is called a *ket vector*, or simply a *ket*.³ The ψ is just a label that we pick for our state while the encasing $|\cdot\rangle$ is there to remind us that this is a vector. Now, superposition tells us that it could happen that the physical system is in a combination of two (or more) states, e.g., we could have something like

$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle, \quad (3.1)$$

for some states $|\psi_1\rangle, |\psi_2\rangle$, and some (complex) numbers α and β . This means that we have much more possibilities than in the classical system. More generally, we can express any vector of a vector space as a linear combination of a set of basis vectors, i.e., we can write

$$|\psi\rangle = \sum_{j=1}^d \psi_j |a_j\rangle, \quad (3.2)$$

where d is the dimension of the vector space and $|a_j\rangle$ a set of basis vectors. As we will see later, the coefficients, ψ_j , in such an expansion are related to the probability of finding the system in state $|a_j\rangle$.

The ket vectors satisfy the ordinary axioms of a vector space. There are two operations, vector addition and scalar multiplication. Under vector addition, the vector space is closed, associative and commutative. This means that for three vectors in the space $|a\rangle, |b\rangle, |c\rangle$, we have

$$\begin{aligned} |a\rangle + |b\rangle &= |c\rangle, & (\text{closed}), \\ (|a\rangle + |b\rangle) + |c\rangle &= |a\rangle + (|b\rangle + |c\rangle), & (\text{associative}), \\ |a\rangle + |b\rangle &= |b\rangle + |a\rangle, & (\text{commutative}). \end{aligned} \quad (3.3)$$

There is a unique identity element of vector addition, which we denote simply by 0, such that

$$|\psi\rangle + 0 = |\psi\rangle. \quad (3.4)$$

The reason why we do not use $|0\rangle$ here is because we want to reserve that notation for something completely different, as we will see later on. There is also a unique vector $(-|\psi\rangle)$ such that

$$|\psi\rangle + (-|\psi\rangle) = 0. \quad (3.5)$$

The vector space is linear and distributive under scalar multiplication. This means that for some complex numbers $z, z_1, z_2 \in \mathbb{C}$,

$$(z_1 + z_2)\psi = z_1|\psi\rangle + z_2|\psi\rangle, \quad z(|\psi\rangle + |\varphi\rangle) = z|\psi\rangle + z|\varphi\rangle. \quad (3.6)$$

Finally, there also exists an identity element with respect to scalar multiplication, i.e., we can multiply with the number 1 and get back the same state, $1|\psi\rangle = |\psi\rangle$.

A basis of a vector space is a minimal set of vectors that spans the space, the number of basis vectors needed gives the dimension of the vector space.

³The notation here (together with the bra vector that we will see in a while, is usually called either the *bra-ket* notation or the Dirac notation, after the physicist Paul Dirac who invented it.

1.3. The dual space and inner product. There is also a corresponding dual vector space. The elements of this space are denoted

$$\langle \varphi |, \quad (3.7)$$

and are called *bra vectors*. The notation and their names becomes slightly more sensical when we introduce the inner product between the bra and the ket, or a bra(c)ket,⁴

$$\langle \varphi | \psi \rangle. \quad (3.8)$$

This is simply a complex number. When we have a finite-dimensional vector space together with an inner product the vector space is a Hilbert space.⁵ Two vectors are said to be orthogonal if their inner product is zero. Furthermore, it is customary to normalize quantum states such that the inner product with itself is equal to one, such vectors are called unit vectors. We will do this automatically, or in other words, we will always set

$$\langle \psi | \psi \rangle = 1. \quad (3.9)$$

Vectors that are both normalized and orthogonal are then called orthonormal. This is, for example, a very good property to demand of a set of basis vectors.

It is often useful to represent the kets as column vectors and the bras as row vectors. We then have the relation

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \leftrightarrow \langle \psi| = (a^* \ b^*), \quad (3.10)$$

and the inner-product (or the bracket) then simply becomes the ordinary multiplication of vectors. We further see that the elements of the corresponding vectors are related by complex conjugation and we have

$$\langle \psi | \psi \rangle = |a|^2 + |b|^2. \quad (3.11)$$

We furthermore have that

$$(\langle \varphi | \psi \rangle)^* = \langle \psi | \varphi \rangle, \quad (3.12)$$

and thus

$$|\langle \varphi | \psi \rangle|^2 = \langle \varphi | \psi \rangle \langle \psi | \varphi \rangle. \quad (3.13)$$

1.4. Composite systems. If we imagine that we have several quantum systems, each in some state represented by some state vector, we can combine the separate system into a combined system using the tensor product of vector spaces, \otimes . If we imagine that we have one system where the state is given by $|\psi\rangle$ and another where the state is given by $|\varphi\rangle$, the state of the composite system is given by

$$|\psi\rangle \otimes |\varphi\rangle. \quad (3.14)$$

This way we can build complicated systems by simply combining several smaller systems. We will see this in action when we discuss quantum circuits. Note that the tensor product does not in general commute.

2. Measurements and probability

2.1. Observables. We have discussed how a quantum state is described by a state vector in a vector space. The quantum state is however not something that we can measure directly.⁶ In fact, it only tells us something about the probability of finding some result upon performing a measurement. Note that this is in stark contrast to the classical case where the state and the outcome of a measurement is for all intents and purposes equal to each other.

⁴Remember that quantum physics was invented long before the invention of the meme, so this was perhaps at the time considered funny. Dirac was also a famously strange man, The strangest man.

⁵When the vector space is infinite-dimensional, some extra subtleties arise, but we will almost exclusively be dealing with finite-dimensional vector spaces in this course.

⁶The actual meaning of the quantum state is something that has spurred a long history of heated discussions. We will discuss some interpretations in the following sections.

Instead, in quantum mechanics, observables are described by linear operators acting on the vector space of states. We say that a linear operator A acts on the states, and denote it by

$$A|\psi\rangle. \quad (3.15)$$

The corresponding action on the bra is given by the *Hermitian conjugate* (or sometimes the adjoint) of A , which we denote by a small dagger

$$A|\psi\rangle \leftrightarrow \langle\psi|A^\dagger. \quad (3.16)$$

When we represent the bras and kets as vectors the operators will be represented by matrices. The action of the dagger is then given by complex conjugation of the elements together with transposition of the matrix. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \quad (3.17)$$

We can construct linear operators through the *outer product*

$$A = |a_1\rangle\langle a_2|. \quad (3.18)$$

Acting with such an operator on a state $|\psi\rangle$ gives

$$A|\psi\rangle = (|a_1\rangle\langle a_2|)|\psi\rangle = \langle a_2|\psi\rangle|a_1\rangle. \quad (3.19)$$

So it transforms the state into the state $|a_1\rangle$ times some complex number.

A very important and useful identity can be derived by considering a complete orthonormal basis $\{|a_j\rangle\}$ and expressing $|\psi\rangle = \sum_j \psi_j|a_j\rangle$, then introduce the operator $A = \sum_j |a_j\rangle\langle a_j|$. We notice that

$$A|\psi\rangle = \left(\sum_j |a_j\rangle\langle a_j| \right) |\psi\rangle = \sum_j |a_j\rangle\langle a_j|\psi\rangle = \sum_j \sum_k \psi_k |a_j\rangle\langle a_j|a_k\rangle = \sum_j \psi_j |a_j\rangle = |\psi\rangle, \quad (3.20)$$

which implies that $\sum_j |a_j\rangle\langle a_j| = \mathbb{1}$, the identity operator on the vector space. This relation is called a completeness relation, or sometimes a resolution of identity, and can be a very useful trick in many computations and proofs in quantum mechanics.

For any observable, say A , there exists a particular set of vectors called the eigenvectors, which we denote $|a\rangle$. They are defined through the relation

$$A|a\rangle = a|a\rangle, \quad (3.21)$$

where a is a complex number called the eigenvalue corresponding to the eigenvector $|a\rangle$ of A . We will typically use the above notation where the eigenvalues and eigenvectors have the same symbol, i.e., the eigenvalue of the eigenvector $|a\rangle$ is given by a . This is standard, and hopefully does not introduce too much confusion.

An especially important class of operators is the class of *Hermitian operators*. They are defined by having the property $A^\dagger = A$. One of the key consequences of this for Hermitian operators is that their eigenvalues are all real. Physical observables in quantum mechanics are always given by Hermitian operators. The reason, as we will see later, is that the result of a measurement in quantum mechanics is given by the eigenvalues of the observable we are measuring. But the results of any physical measurement should of course be a real number, so that we should impose that the observables are Hermitian operators. Another important property of Hermitian operators is that their eigenvectors form a complete set, i.e., any state can be expressed in the eigenvectors. Note however, that if the eigenvalues are the same the eigenvectors need not be orthogonal.

An operator A is called *normal* if it satisfies $A^\dagger A = A A^\dagger$. Such operators satisfies an important theorem called the spectral decomposition theorem. It states that an operator is normal if and only if it is diagonalizable with respect to some basis. This means that we can always express a normal operator, A , as $A = \sum_j a_j |j\rangle\langle j|$, where a_j are the eigenvalues of A and $|j\rangle$ an orthonormal basis where each vector is also an eigenvector of A (with eigenvalue a_j). Obviously, Hermitian operators are always normal.

Suppose now that we have two different observables A and B and we want to know if we can express them both in terms of the same basis. Or in other words, if we can write $A = \sum_j a_j |j\rangle\langle j|$ and $B = \sum_j b_j |j\rangle\langle j|$.

If this is possible we say that A and B are simultaneously diagonalizable. It turns out that this can only be done if A and B commute with each other, that is, if and only if

$$[A, B] := AB - BA = 0. \quad (3.22)$$

We can again use the tensor product to build larger system. If we have a system that is a composite system of say two different vector spaces as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle, \quad (3.23)$$

we can build composite operators acting on this tensor product as

$$A = A_1 \otimes A_2, \quad A\psi = A_1|\psi_1\rangle \otimes A_2|\psi_2\rangle. \quad (3.24)$$

2.2. The wave function. Consider now a complete set of commuting observables, A, B, C, \dots together with an orthonormal basis $|a, b, c, \dots\rangle$, where a, b, c, \dots are the corresponding eigenvalues of the observables. An arbitrary state $|\psi\rangle$ can then be expanded in this basis as

$$|\psi\rangle = \sum_{a,b,c,\dots} \psi(a, b, c, \dots) |a, b, c, \dots\rangle. \quad (3.25)$$

The set of coefficients,

$$\{\psi(a, b, c, \dots) = \langle a, b, c, \dots | \psi \rangle\}, \quad (3.26)$$

is called the *wave function* of the system in the a, b, c, \dots basis. The individual coefficients $\psi(a, b, c, \dots)$ are also called the *probability amplitude* for finding the system in the $|a, b, c, \dots\rangle$ state, or sometimes just the amplitude. It is important to note that this is not the same as the probability as we will see next. For one thing, it is in general a complex number. The actual probability of finding the eigenvalues corresponding to $|a, b, c, \dots\rangle$, is instead given by the absolute value squared of $\psi(a, b, c, \dots)$.

2.3. Measurements. The idea of a measurement in quantum mechanics is that we measure some observable A and the outcome will be an eigenvalue of A , where the corresponding probability of getting this result is captured by the coefficient of the state when expanded in the eigenvectors of the measured observable.

In other words, we start with an observable A that we want to measure. We express it in its complete basis of eigenvectors $A = \sum_j a_j |a_j\rangle\langle a_j|$. Then we have some system $|\psi\rangle$ which we want to measure the observable in. So we expand $|\psi\rangle = \sum_j \psi_j |a_j\rangle$. The measurement will then return an eigenvalue of A , let us say a_j , and the probability of getting this result is given by $|\psi_j|^2$. Note that we have enforced the normalization

$$1 = \langle \psi | \psi \rangle = \sum_j |\psi_j|^2, \quad (3.27)$$

so this interpretation as a probability makes sense.

After the measurement, the system has “collapsed” to the state $|a_j\rangle$ and we can measure A again to find the same result, a_j .

The expectation value of an observable A in the state $|\psi\rangle$ is denoted $\langle A \rangle_\psi$ and defined by

$$\langle A \rangle_\psi := \langle \psi | A | \psi \rangle = \sum_j a_j |\langle \psi | a_j \rangle|^2, \quad (3.28)$$

where $|a_j\rangle$ is the complete set of eigenvectors of A . Note that, with the above interpretation this is simply the ordinary definition of the expectation value.

As just mentioned, if we measure an observable A and find that the system is in state $|a_1\rangle$, say, then if we make another measurement asking if the system is in state $|a_1\rangle$ we will get a positive answer with probability one. So this perhaps feels like nothing special is going on, why not just say that the system was in state $|a_1\rangle$ all along?

Well, the tricky thing with quantum mechanics is that if we now measure another observable that is not commuting with A , say B , and find the result $|b_1\rangle$, and afterwards return to measure A again, it is no

longer true that we are certain to find the result a_1 . We are basically back at square one and the only thing we can say is that there is a probability $|\langle b_1|a_1\rangle|^2$ to find the result a_1 .

To see this a bit more clearly, we consider a simple two-level system. This means that we have a two-dimensional vector space.⁷ We introduce an orthonormal basis

$$\{|u\rangle, |d\rangle\}, \quad (3.29)$$

such that we can express any state as

$$|\psi\rangle = \alpha|u\rangle + \beta|d\rangle, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (3.30)$$

Next, we introduce an observable σ_z defined by

$$\sigma_z|u\rangle = |u\rangle, \quad \sigma_z|d\rangle = -|d\rangle. \quad (3.31)$$

I.e., the basis vectors are eigenvectors of σ_z with eigenvalues ± 1 , respectively.

We now measure σ_z and get some result. Let us assume that this is $+1$, and the state collapses to $|u\rangle$. As said before, we can now measure σ_z again and again and every time we will get the result $+1$.

But, there is nothing special with the basis defined by $|u\rangle$ and $|d\rangle$, we can easily well pick another basis. For example,⁸

$$|l\rangle := \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle), \quad |r\rangle := \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle). \quad (3.32)$$

Related to this basis we can introduce a new observable, σ_x , that has these vectors as eigenvectors,

$$\sigma_x|l\rangle = |l\rangle, \quad \sigma_x|r\rangle = -|r\rangle, \quad (3.33)$$

and which does not commute with σ_z . If we now measure σ_x in our system, which has collapsed to $|u\rangle$ after the first measurement, we will get the results ± 1 with probabilities

$$\begin{aligned} |\langle u|l\rangle|^2 &= \frac{1}{2}|\langle u|(|u\rangle + |d\rangle)|^2 = \frac{1}{2}, \\ |\langle u|r\rangle|^2 &= \frac{1}{2}|\langle u|(|u\rangle - |d\rangle)|^2 = \frac{1}{2}. \end{aligned} \quad (3.34)$$

Let us again assume that the result is $+1$ such that the state collapses to $|l\rangle$. Now you might start to see the problem. If we return to measure σ_z , we will no longer find $+1$ with probability one but instead we have

$$\begin{aligned} |\langle l|u\rangle|^2 &= \frac{1}{2}, \\ |\langle l|d\rangle|^2 &= \frac{1}{2}. \end{aligned} \quad (3.35)$$

The two outcomes are equally probable. This is part of the mysterious and indeterministic nature of quantum mechanics. This uncertainty in measuring non-commuting observables is captured by the famous *Heisenberg's uncertainty principle*.

It is, perhaps, easy to see that, if the observables do commute we can measure them simultaneously. Since we can then diagonalize them in the same basis.

Let us note an important fact. If we have two states that only differ by an overall phase, say, $|\psi\rangle$ and $|\varphi\rangle = e^{i\gamma}|\psi\rangle$, then the statistical properties of these states are the same. This is easily seen from the fact that we have

$$|\varphi\rangle = e^{i\gamma}|\psi\rangle \implies \langle\varphi| = \langle\psi|e^{-i\gamma}, \quad (3.36)$$

and we thus have

$$\langle\varphi|\varphi\rangle = \langle\psi|e^{-i\gamma}e^{i\gamma}|\psi\rangle = \langle\psi|\psi\rangle = \sum_j |\psi_j|^2. \quad (3.37)$$

For this reason, in quantum mechanics, we do not distinguish between states that differ only by an overall phase.

⁷This kind of system will of course be the main protagonist of this course, since the qubit is a two-level system. But for now we simply think of it in slightly more abstract terms.

⁸as an exercise you can show that this is an orthonormal set of vectors,

3. Evolution

3.1. Unitary operators. The evolution of a quantum state is described by a special kind of operator, namely a unitary operator. These are operators that have the property that their inverse is equal to their adjoint, or, in other words, $U^\dagger U = U U^\dagger = \mathbb{1}$.⁹

To understand this statement better, we consider a system that at some time t is in the state $|\psi(t)\rangle$. Now, we want to study how this system changes over time. For simplicity, we start at $t = 0$ and ask how is the state $|\psi(t)\rangle$ related to the state $|\psi(0)\rangle$? We encode this change in an operator that we call $U(t)$ such that we have

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle. \quad (3.38)$$

Now, to be able to say something more about this mysterious operator $U(t)$ we want to introduce some restrictions. First of all, we want to demand that it is linear. This is natural from what we have discussed before. Quantum operators are typically linear. Less trivial is the statement that we want to enforce the operator to preserve distinguishability. This means that, if we have two orthogonal states, such that they are distinguishable by a measurement, we want them to still be orthogonal after the time evolution.

If we pick two elements $|a_j\rangle$ and $|a_k\rangle$ of an orthonormal basis to represent the states at $t = 0$, we have the condition

$$\langle a_j | a_k \rangle = \delta_{jk}, \quad (3.39)$$

where δ_{jk} is the Kronecker symbol.¹⁰ But if we now let them evolve in time using $U(t)$ we want to have

$$\langle a_j | U^\dagger(t) U(t) | a_k \rangle = \delta_{jk}, \quad (3.40)$$

and we see that $U^\dagger(t)U(t)$ acts as the unit operator. From this you can prove that the same is true for the action on any states. We thus need the time evolution operator to satisfy $U^\dagger(t)U(t) = \mathbb{1}$. This is exactly the definition of a unitary operator. So, time evolution is described by a unitary operator.

3.2. The Schrödinger equation. A very important operator in quantum mechanics is the Hamilton operator, or sometimes just the Hamiltonian. This is the operator corresponding to the observable of energy. The Hamiltonian determines the evolution of the system through the Schrödinger equation¹¹

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle. \quad (3.41)$$

More specifically, this is called the time-dependent Schrödinger equation. Here \hbar (pronounced *h-bar*) is the famous Planck's constant, $\hbar \sim 1.0546 \times 10^{-34} \text{ kg m}^2/\text{s}$.¹²

The Hamiltonian is Hermitian and we can expand it in its complete set of eigenvectors.

$$H = \sum_j E_j |E_j\rangle \langle E_j|. \quad (3.42)$$

These eigenstates are called the energy eigenstates and the corresponding eigenvalues are the results of a measurement of the energy of the system. Since the $|E_j\rangle$ are eigenstates of the Hamiltonian we have

$$H|E_j\rangle = E_j|E_j\rangle, \quad (3.43)$$

which is sometimes called the time-independent Schrödinger equation.

By solving Schrödinger's equation, we find the connection to the operator $U(t)$ discussed earlier. Namely,

$$U(t) = e^{-\frac{i}{\hbar} H(t)}. \quad (3.44)$$

⁹In quantum circuits, the gates are always unitary operators as you will see throughout the course.

¹⁰ $\delta_{jk} = 1$ if $j = k$ and 0 if $j \neq k$.

¹¹Named after its inventor, the cat-friendly Austrian Erwin Schrödinger.

¹²The German physicist Max Planck was the person who, sort of by mistake, started the whole field of quantum physics. He introduced a constant which he called h , which was later divided by 2π to give the constant $\hbar := \frac{h}{2\pi}$, which we now call Planck's constant.

3.3. A note on (in)determinism. As we have mentioned already in the introduction, and seen in the discussion of measurements, quantum mechanics is inherently non-deterministic. But the discussion of time evolution of the quantum state looks very deterministic, right? This is true. The time evolution of the quantum state is a deterministic process, but this does not necessarily mean that quantum mechanics is deterministic.

In classical physics, making measurements does not affect the system and the result of a measurement is equivalent to the state of the system, both before and after the measurement. This is the basis of the determinism in classical physics. By knowing the state and knowing the equations of motion, we can determine where the state came from and where it is going. As we have seen, this is no longer true in quantum physics. Time evolution of the quantum state is deterministic, but knowing the state does not tell you with certainty the result of a general measurement.

4. Summary: Quantum postulates

We can now summarize what we have learned so far into four postulates of quantum mechanics.

- States are described by unit vectors in a complex vector space (in fact a Hilbert space), and observables are described by linear Hermitian operators.
- The possible outcomes of a measurement are given by the eigenvalues of the operator corresponding to the observable being measured.
- If the system is in a state $|\psi\rangle$, and we measure an observable A with eigenvectors $|a_j\rangle$ and eigenvalues a_j , the probability of measuring eigenvalue a_j is given by

$$P(a_j) = |\langle a_j | \psi \rangle|^2 = \langle \psi | a_j \rangle \langle a_j | \psi \rangle. \quad (3.45)$$

- The evolution of a quantum system is described by unitary operators.

5. What actually is the quantum state?

At this point, you might be asking yourself what the meaning of the quantum state is. After all, measurements tells us that eventually the state will not be in a superposition, the thing we observe is a definitive classical state, so how do we know that the state was ever in a superposition of other states? Well, such questions have given rise to a large number of debates on the interpretation of quantum mechanics.¹³

Many early interpretations of quantum mechanics involved hidden variables, i.e., that there are some hidden variables that we do not know about which determines the measurements in a deterministic fashion. These have been essentially refuted by a number of results, such as Gleason's theorem and variations thereof. These type of results typically go under the name of Bell's theorem, and in principle they rule out almost all hidden variables theories. The experimental verification of these results was the subject of the Nobel prize in physics 2022.¹⁴

In contrast, the widely considered Copenhagen interpretation of quantum mechanics (with variations) is essentially Bayesian. In this interpretation, the nature of quantum mechanics is essentially non-deterministic, and we should not require one to consider the "exponential" dimension of the quantum state prior to measurement any more so than we require a person throwing a die to consider the probability distribution over the outcomes. Measurements give rise to a (practically) irreversible process in which the state is affected.

Another famous interpretation is the many-worlds interpretation due to Hugh Everett. Here, time is considered as a tree, having many branches and each branch corresponds to a possible result of a measurement. This gives rise to an uncountable number of worlds or universes. The many-worlds

¹³See for example Wikipedia

¹⁴Awarded to the three experimentalists Alain Aspect, John Clauser and Anton Zeilinger.

interpretation is thus inherently deterministic, as the universal wave function never collapses to one particular state.

Finally, let us mention the de Broglie-Bohm interpretation. This is a kind of hidden variables theory where the problems of Bell's theorem are circumvented by embracing non-locality. It is thus a deterministic theory and particles have a definite configuration at all times, even when not observed.