Motion planning: combinatorial path planning

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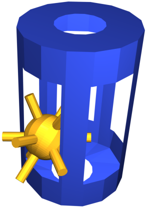
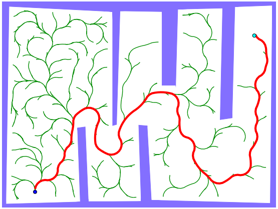
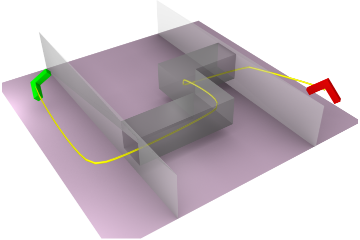






The art of motion planning

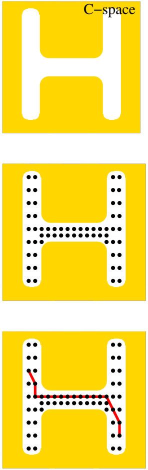
Continuous space

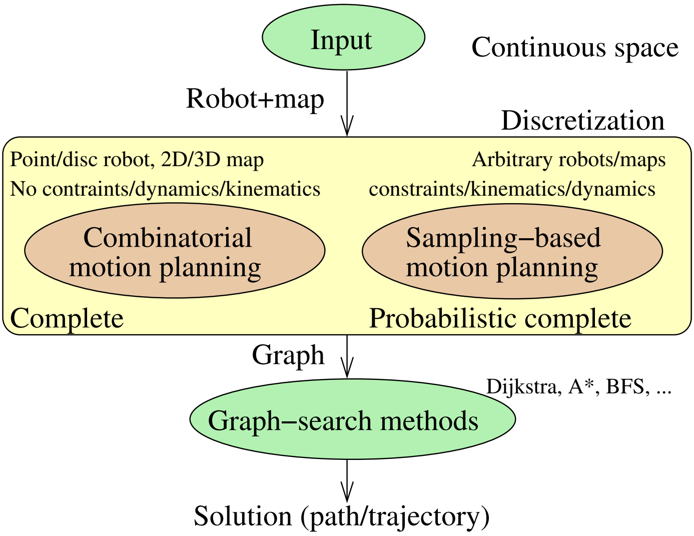
↓

Discretization

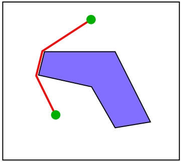
↓

Search



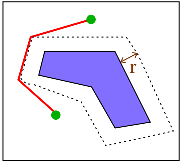

Input
Continuous space
Robot+map
Discretization
Point/disc robot, 2D/3D map
No contraints/dynamics/kinematics
Arbitrary robots/maps
constraints/kinematics/dynamics
Combinatorial motion planning
Sampling−based motion planning
Complete
Probabilistic complete
Graph
Dijkstra, A*, BFS, ...
Graph−search methods
Solution (path/trajectory)

- Assume point/disc robots
- Use geometric (usually polygonal) representation of W
- In these cases, representation of $\mathcal{W}$ is also representation of $\mathcal{C}$
- The representation is explicit $\rightarrow$ enumeration of obstacles is easy
- Voronoi diagram, Visibility map, Decomposition-based methods



Point robot in 2D or 3D $\mathcal{W}$

- The map of $\mathcal{W}$ is also representation of $\mathcal{C}$
- Polygons/polyhedrons are suitable

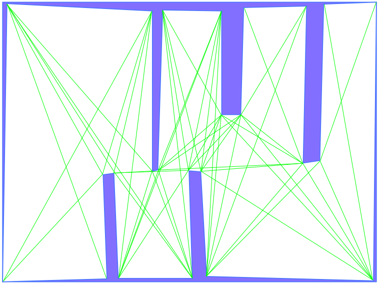


Disc/sphere robot in 2D or 3D $\mathcal{W}$

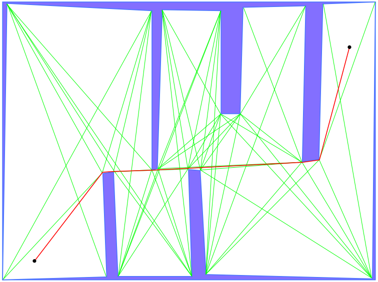
- The obstacles are “enlarged” by radius of the robot (Minkowski sum)
- Then, representation of $\mathcal{W}$ is also representation of $\mathcal{C}$

Visibility graph

- Two points v_i, v_j are visible $\iff$ $(sv_i + (1-s)v_j) \in \mathcal{C}_{\text{free}}, \quad s \in (0,1)$
- Visibility graph (V, E), V are vertices of polygons, E are edges between visible points
- Start/goal are connected in same manner to visible vertices



Visibility graph



After connecting start/goal + path

- No clearance
- Suitable only for 2D

Visibility graph (VG)

- Straightforward, näive implementation $O(n^3)$

Input: polygonal obstacle
Output: visibility graph $G = (V, E)$

```
V = all vertices of polygonal obstacles
foreach u, v ∈ V do
    foreach obstacle edge e do
        if segment u, v intersects e then
            continue;
        add edge u, v to E
```

- n^2 pairs of vertices
- Complexity of checking one intersection is $O(n)$

$\rightarrow$ Total complexity $O(n^3)$

Fast methods

- Lee's algorithm $O(n^2 \log n)$
- Overmars/Welz method $O(n^2)$
- Ghosh/Mount method $O(|E| n \log n)$

☛ Lee, Der-Tsai, Proximity and reachability in the plane, 1978

☛ D. Coleman, Lee's O(n2 log n) Visibility Graph Algorithm Implementation and Analysis, 2012.

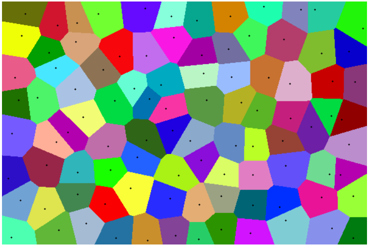
☛ M. H. Overmars, E. Welzl, New methods for Computing Visibility Graphs, Proc. of 4th Annual Symposium on Comp. Geometry, 1998

☛ S. Ghosh and D. M. Mount, An output-sensitive algorithm for computing visibility graphs, SIAM Journal on Computing, 1991

Voronoi diagram

- Let $P = v_1, \ldots, v_n$ are n distinct points ("input sites") in a $d-$dimensional space
- Voronoi Diagram (VD) divides P into n cells $V(p_i)$

$$V(p_i) = \{x \in \mathbf{R}^d : ||x - p_i|| \leq ||x - p_j|| \;\; \forall j \leq n\}$$



- Cells are convex
- Used in point location (1-nn search), closest-pair search, spatial analysis
- Construction using Fortune's method in $O(n \log n)$

☛ S. Fortune. A sweepline algorithm for Voronoi diagrams. Proc. of the 2nd annual composium on Computational geometry. pages 313-322. 1986.

Voronoi diagram

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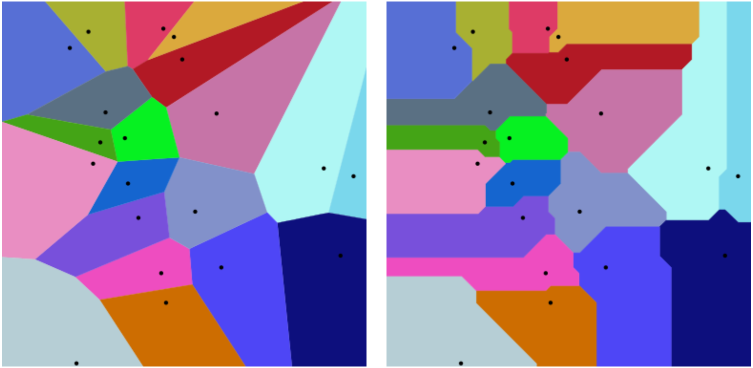
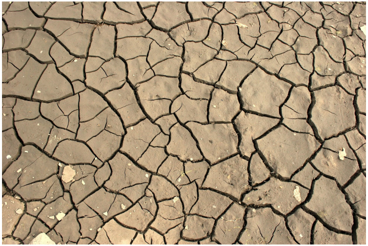
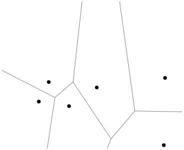
- Note, that other metrics can be considered



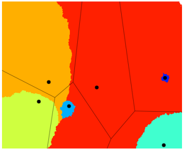
Voronoi diagrams are everywhere


- (Basic) Voronoi diagram: computed on points
- Generalized Voronoi Diagram: computed on e.g., points + weights, segments, spheres, . . .



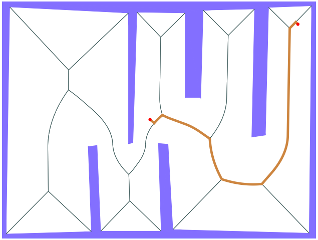
Classic VD

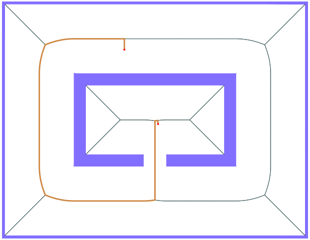


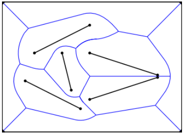
Weighted VD

Segment Voronoi Diagram (SVD)

- computed on line-segments describing obstacles
- requires polygonal map or line/segment map
- ✓ Maximal clearance
  - largest distance between a path and the nearest obstacle
- Is it optimal? Is it complete?





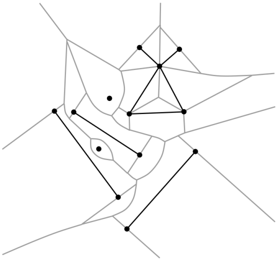


Segment VD

Segment Voronoi diagram: complexity

Algorithms for computing Segment Voronoi diagram of n segments

- Lee & Drysdale: $O(n \log^2 n)$, no intersections
- Karavelas: $O((n+m)\log^2 n)$, m intersections between segments



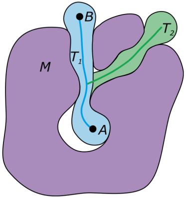
Karavelas 2004

☛ Karavelas, M. I. "A robust and efficient implementation for the segment Voronoi diagram." International symposium on Voronoi diagrams in science and engineering. 2004

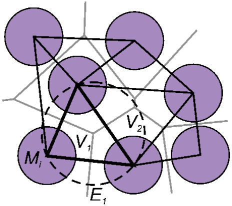
☛ Lee, D. T, R. L. Drysdale, III. "Generalization of Voronoi diagrams in the plane." SIAM Journal on Computing 10.1 (1981): 73-87.

Voronoi diagrams in bioinformatics

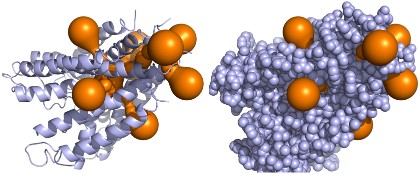
- Proteins are modeled using hard-sphere model
- Weighted Voronoi diagram of the spheres (weight is the atom radii — Van der Waals radii)
- Path in the Voronoi diagram reveals “void space” and “tunnels”
- Tunnel properties (e.g. bottleneck) estimate possibility of interaction between protein and a ligand



Tunnels*

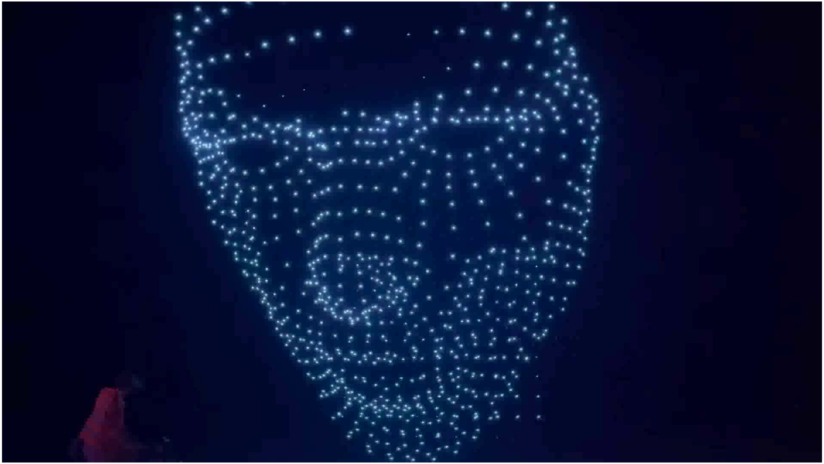


Voronoi diagram *


Tunnels on 1BL8

* ☛ A. Pavelka, E. Sebestova, B. Kozlikova, J. Brezovsky, J. Sochor, J. Damborsky, CAVER: Algorithms for Analyzing Dynamics of Tunnels in Macromolecules, IEEE/ACM Trans. on compt. biology and bioinformatics, 13(3), 2016.

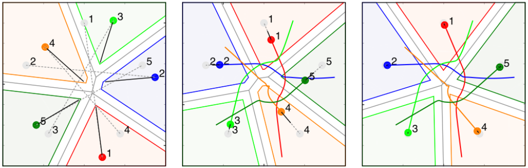
- Change of positions between various formations (e.g. in drone art)



www.youtube.com/watch?v=YH1BD7kKqKw

Voronoi diagram for collision avoidance

- Change of positions between various formations (e.g. in drone art)

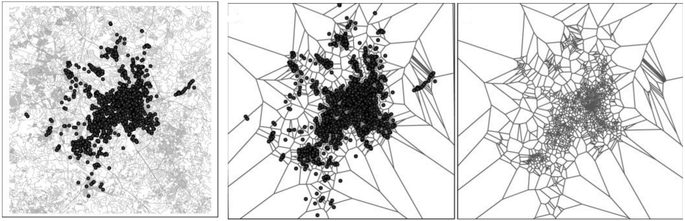


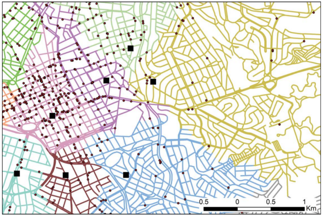
(a) (b) (c)

☛ Zhou, Dingjiang, Zijian Wang, Saptarshi Bandyopadhyay, and Mac Schwager. Fast, On-Line Collision Avoidance for Dynamic Vehicles Using Buffered Voronoi Cells. IEEE Robotics and Automation Letters, (2), 2017.

Voronoi diagram for spatial analysis

- One of first analysis was Cholera epidemic in London
- Often used in criminology

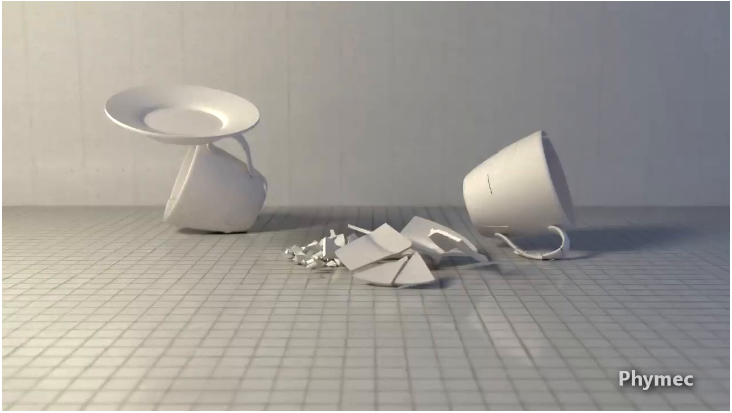




☛ Melo, S. N. D., Frank, R., Brantingham, P. (2017). Voronoi diagrams and spatial analysis of crime. The Professional Geographer, 69(4), 579-590.

Voronoi diagram in computer graphics

- Used in many low-level routines (e.g., point location)
- Modeling fractures
  - Object is filled with some random points
  - VD is computed to provide set of convex cells
  - Interaction between cells can be modeled e.g. using rigid body dynamics



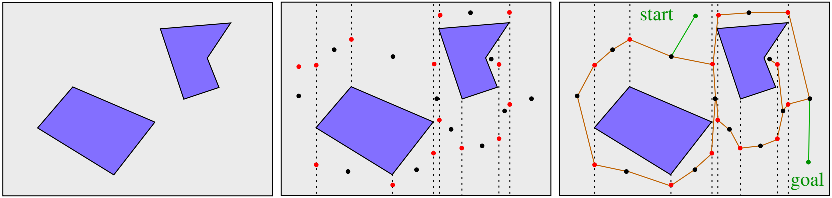
www.youtube.com/watch?v=FIPu9_OGFgc

Decomposition-based methods

- The free space is partitioned into a finite set of cell
  - Determination of cell containing a point should be trivial
  - Computing paths inside the cells should be trivial
- The relations between the cells is described by a graph

Vertical cell decomposition

- Make vertical line from each vertex, stop at obstacles
- Determine centroids of the cells, centers of each segments
- Graph connects the neighbor centroids through the centers
- Connect start/goal to centroid of their cells
- Can be built in $O(n \log n)$ time

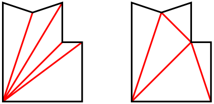


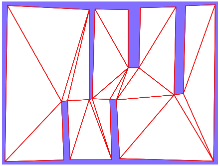
Decomposition via triangulation I

- Variant of decomposition-based methods
- $\mathcal{C}_{\text{free}}$ is triangulated
- Can be computed in $O(n \log \log n)$ time
- Polygons can be triangulated in many ways
- $\mathcal{C}_{\text{free}}$ is represented by graph $G = (V, E)$
  - V are centroids of the triangles
  - $E = (e_{i,j})$ if Δ_i is neighbor of Δ_j

  Or

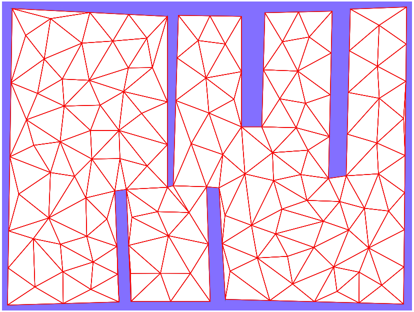
  - V are vertices of the triangulation
  - E are edges of the triangulation
- Planning: start/goal are connected to graph, then graph search



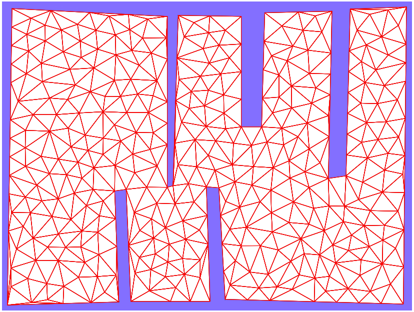


Decomposition via triangulation II

- Finer triangulation via Constrained Delaunay Triangulation (CDT)
  - if a triangle does not meet a criteria, it is further triangulated
  - criteria: triangle area or the largest angle



CDT



Finer CDT (area of Δ)

Decomposition via triangulation II

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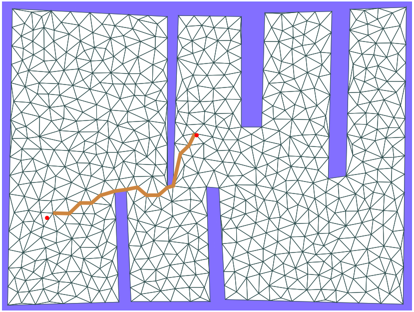
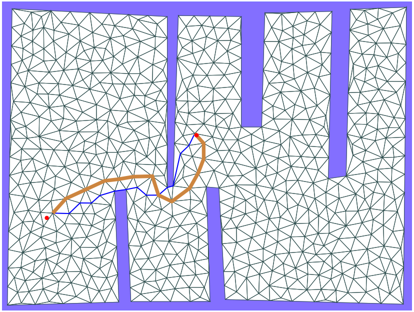
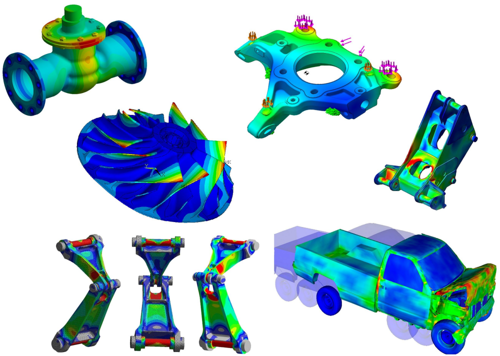


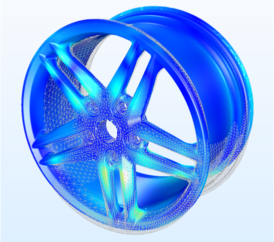
Path on edges

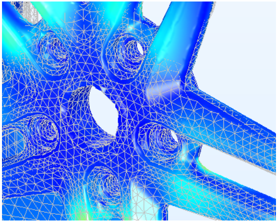


Modification: ignore segments connecting obstacles

- Structural analysis: modeling behavior of a structure under load, wind, pressure, . . .
- Finite element method







Navigation functions

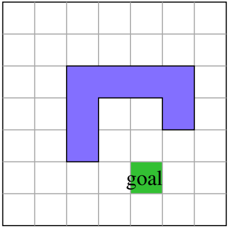
- Let's assume a forward motion model

$$\dot{q} = f(q, u)$$

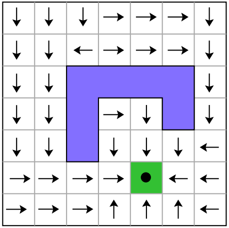
where $q \in \mathcal{C}$ and $u \in \mathcal{U}$; $\mathcal{U}$ is the action space

- The navigation function $F(q)$ tells which action to take at q to reach the goal

Example: robot moving on grid, actions $\mathcal{U} = \{\rightarrow, \leftarrow, \uparrow, \downarrow, \bullet\}$



Discrete planning problem



Navigation function

- In discrete space, navigation f. is a by-product of graph-search methods

Wavefront planner

- Simple way to compute navigation function on discrete space X
- Explores X in "waves" starting from goal until all states are explored

```
open = {goal}
i = 0
while open ≠ ∅ do
    wave = ∅          // new wave
    foreach x ∈ open do
        value(x) = i
        foreach y ∈ N(x) do
            if y is not explored
              then
                add y to wave
    i = i + 1
    open = wave
```

- $N(x)$ are neighbors of x
- 4-/8-point connectivity
- The increase of the wave value i should reflect the distance between x and its neighbors
- Path is retrieved by gradient descend from start
- $O(n)$ time for n reachable states

		∞	∞	∞	∞	
		∞			∞	
		∞				
				0		

goal state

		∞	∞	∞	∞	
		∞			∞	
		∞	1	1	1	
			1	0	1	
			1	1	1	

$i = 1$

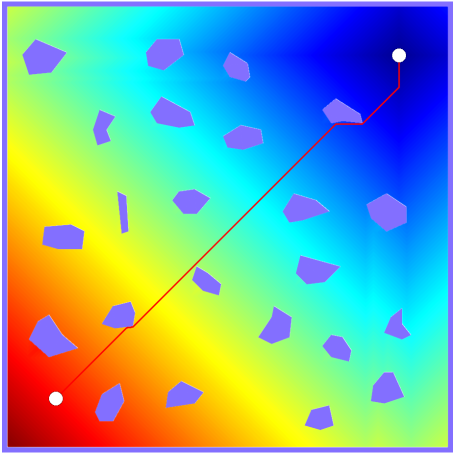
		∞	∞	∞	∞	
		∞	2	2	∞	2
		∞	1	1	1	2
		2	1	0	1	2
		2	1	1	1	2

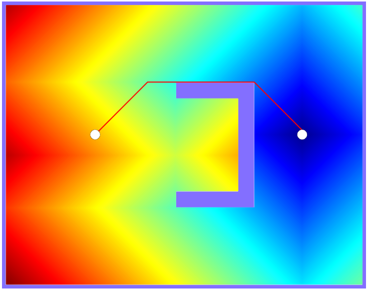
$i = 2$

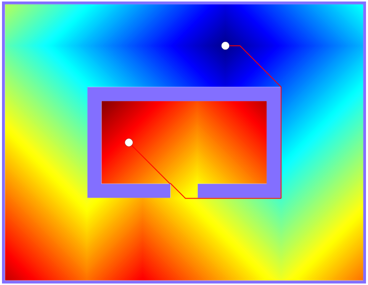
7	7	7	6	5	5	5
6	6	6	6	5	4	4
5	5	∞	∞	∞	∞	3
4	4	∞	2	2	∞	2
4	3	∞	1	1	1	2
4	3	2	1	0	1	2
4	3	2	1	1	1	2

$i = 7$

Wavefront planner

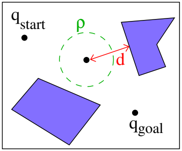






Potential field: principle

- Potential field U: the robot is repelled by obstacles and attracted by q_{goal}
- Attractive potential U_{att}, repulsive potential U_{rep}
- Weights K_{att} and K_{rep}, d is the distance to the nearest obstacle, ϱ is radius of influence



$$U_{att}(q) = \frac{1}{2} K_{att} dist(q, q_{\mathrm{goal}})^2 \qquad U_{rep}(q) = \begin{cases} \frac{1}{2} K_{rep}(1/d - 1/\varrho)^2 & \text{if } d \leq \varrho \\ 0 & \text{otherwise} \end{cases}$$

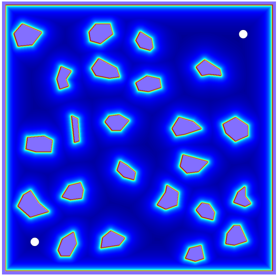
- Combined attractive/repulsive potential

$$U(q) = U_{att}(q) + U_{rep}(q)$$

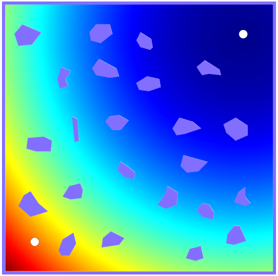
- Goal is reached by following negative gradient $-\nabla U(q)$
- Gradient-descend method

☛ Y. K. Hwang and N. Ahuja, A potential field approach to path planning, IEEE Transaction on Robotics and Automation, 8(1), 1992.

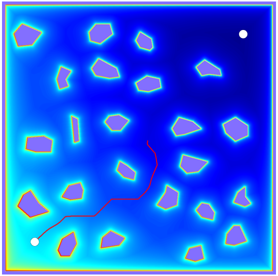
Potential field: parameters



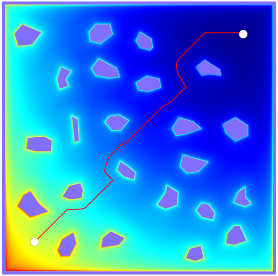
$K_{att} = 0$, no attraction



$K_{att} \gg K_{rep}$, no repulsion



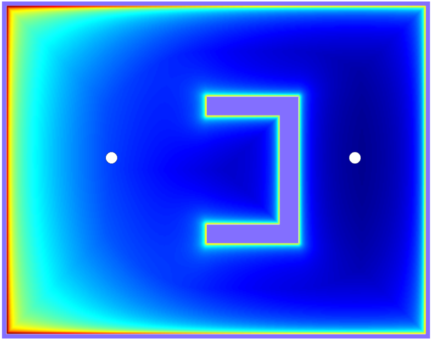
$K_{att} \sim K_{rep}$



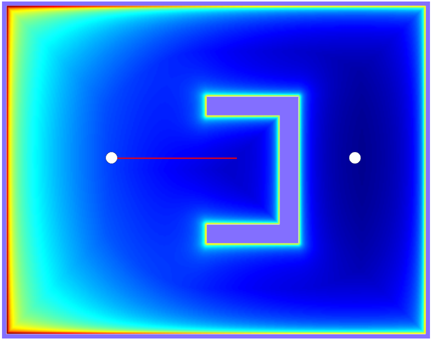
optimal settings

Potential field: local minima problem

- Potential field may have more local minima/maxima
- Gradient-descent stucks there



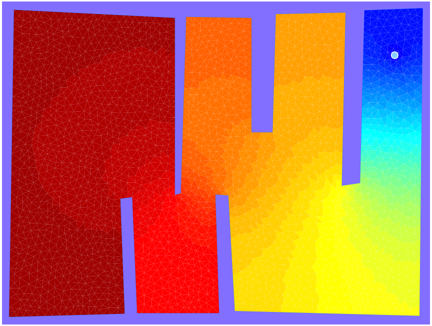
potential field



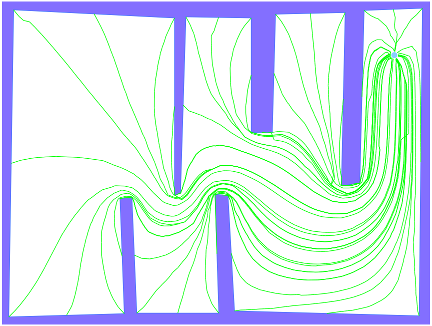
gradient-descent to minimum

- Escape using random walks
- Use a better potential function without multiple local minima — harmonic field

- Harmonic field is an ideal potential function: only one extreme



Harmonic field



Paths from various q_{init}

Images by J. Mačák, Multi-robotic cooperative inspection, Master thesis, 2009

Potential field: summary

- Usually computed using grid or a triangulation of the $\mathcal{W}$
- Suitable for 2D/3D $\mathcal{C}$-space
    - memory requirements (in case of grid-based computation)
    - requires to compute distance d to the nearest obstacle in $\mathcal{C}$!
- Parameters K_{att}, K_{rep} and ϱ need to be tuned
- Problem with local minima $\rightarrow$ harmonic fields

But how to really find the path?

So far we know . . .

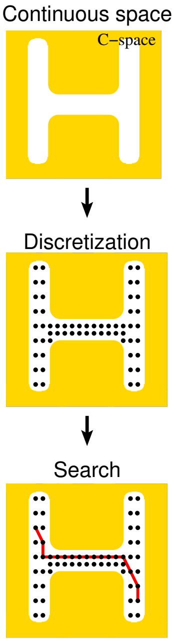
- Visibility graphs, Voronoi diagrams, Decomposition-based planners
- Navigation functions & Potential fields

What they do?

- Discretize workspace/$\mathcal{C}$-space by “converting” it to a graph structure
- The graph is also called **roadmap**
- The roadmap is a “discrete image” of the continuous $\mathcal{C}$-space
- The path is then found as path in the graph

Graph-search

- Breath-first search
- Dijkstra
- A*, D* (and their variants)



Graph search: Dijkstra's algorithm

- Finds shortest path from $s \in V$ (source) to all nodes
- dist(v) is the distance traveled from the source to the node s; prev(v) denotes the predecessor of node v

```
Q = ∅
for v ∈ V do
    prev[v] = -1          // predecessor of v
    dist[v] = ∞               // distance to v
dist[s] = 0
add all v ∈ V to Q
while Q is not empty do
    u = vertex from Q with min dist[u]
    remove u from Q
    foreach neighbor v of u do
        dv = dist[u] + d_{u,v}
        if dv < dist[v] then
            dist[v] = dv
            prev[v] = u
```

- Path from $v \rightarrow s$: $\quad v, pred[v], pred[pred[v]], \ldots s$

☛ Dijkstra, E. W. "A note on two problems in connection with graphs." Numerische mathematik 1.1 (1959): 269-271.

Completeness and optimality

Visibility graph

- Complete and optimal

Voronoi diagram, decomposition-based method

- Complete, non-optimal

Navigation function

- Complete
- Optimal for Wavefront/Dijkstra/-based navigation functions

Potential field

- Complete only if harmonic field is used (one local minima!)

Consider the limits of these methods!

- Point/Disc robots, low-dimensional $\mathcal{C}$-space

☛ E. Rimon and D. Koditschek. "Exact robot navigation using artificial potential functions." IEEE Transactions on Robotics and Automation, 1992.

Do we always need optimal solution?

- No! in many cases, non-optimal solution is fine
  - e.g. for assembly/disassembly studies, computational biology
  - generally: if the **existence of a solution** is enough for subsequent decisions
- in industry:
  - scenarios, where robot waits due to mandatory technological breaks
  - e.g., in robotic welding and painting

When to prefer optimal one?

- Repetitive executing of the same plan
- Benchmarking of algorithms

It is necessary to carefully design the criteria!

Shortest path vs. fastest path vs. path for good spraying

Summary of the lecture

- Motion planning: how to move objects and avoid obstacles
- Configuration space $\mathcal{C}$
- Generally, planning leads to search in continuous $\mathcal{C}$
- But we (generally) don't have explicit representation of $\mathcal{C}$
- We have to first create a discrete representation of $\mathcal{C}$
- and search it by graph-search methods
- Special cases: point robot and 2D/3D worlds
  - Explicit representation of $\mathcal{W}$ is also rep. of $\mathcal{C}$
  - Geometric planning methods: Visibility graph, Voronoi diagram, decomposition-based
  - Also navigation functions + potential field