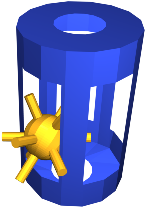
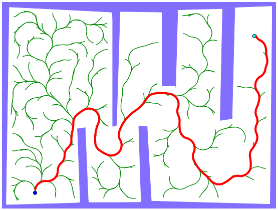
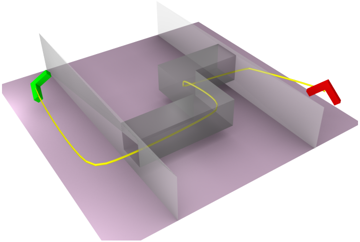
Motion planning: basic concepts

Vojtěch Vonásek

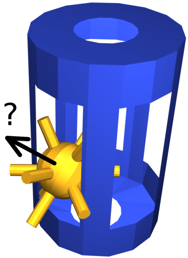
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague



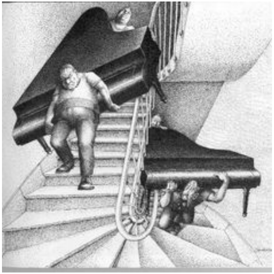




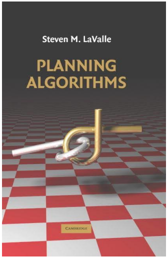
Informal definition: Motion planning is about automatic finding of ways how to move an object (robot) while avoiding obstacles (and considering other constraints).

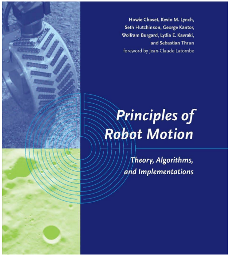


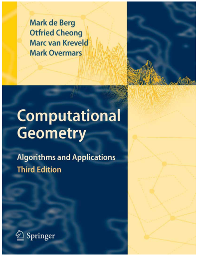
- “Piano mover’s problem”
- Classical problem of robotics
- Relation to other fields
  - Mathematics: graph theory & topology
  - Computational geometry: collision detection
  - Computer graphics: visualizations
  - Control theory: feedback controllers required to navigate along paths
- Motion planning finds application in many practical tasks



References







- S. M. LaValle, Planning algorithms, Cambridge, 2006, online: `planning.cs.uiuc.edu`
- H. Choset, K. M. Lynch et al., Principles of Robot Motion: Theory, Algorithms, and Implementations (Intelligent Robotics and Autonomous Agents series), Bradford Book, 2005
- M. de Berg, Computational Geometry: Algorithms and Applications, 1997
- C. Ericson. Real-time collision detection. CRC Press, 2004.

Introduction & motivation

↓

Formal definition, configuration space
Why we need discretization of configuration space

↓ ↓

Combinatorial planning
(Low-dimensional cases)
Visibility graphs, Voronoi diagrams, . . .

Sampling-based planning
(High-dimensional cases)
RRT, PRM, EST, . . .

Technical details
benchmarking
sampling, collision-detection, metrics,
planning under constraints, physical simulations, tips & tricks, . . .

World $\mathcal{W}$

- is space where the robot operates
- $\mathcal{W}$ is usually $\mathcal{W} \subseteq \mathbf{R}^2$ or $\mathcal{W} \subseteq \mathbf{R}^3$
- $\mathcal{O} \subseteq \mathcal{W}$ are obstacles

Robot $\mathcal{A}$

- $\mathcal{A}$ is the geometry of the robot
- $\mathcal{A} \subseteq \mathbf{R}^2$ (or $\mathcal{A} \subseteq \mathbf{R}^3$)
- or set of links $\mathcal{A}_1, \dots \mathcal{A}_n$ for $n-$body robot

Configuration q

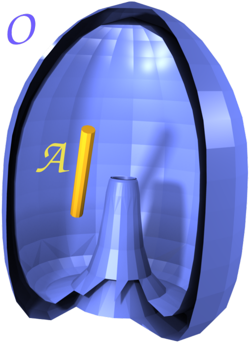
- Specifies position of **every** point of $\mathcal{A}$ in $\mathcal{W}$
- Usually a vector of **Degrees of freedom (DOF)**

$$q = (q_1, q_2, \dots, q_n)$$

Configuration space $\mathcal{C}$ (aka C-Space or $\mathcal{C}$-space)

- $\mathcal{C}$ is a set of **all** possible configurations

3D Bugtrap benchmark

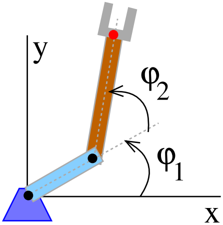


$\mathcal{W} \subseteq \mathbf{R}^3$, $\mathcal{A} \subseteq \mathbf{R}^3$
$\mathcal{O} \subseteq \mathbf{R}^3$
(x, y, z) is 3D position
(r_x, r_y, r_z) is 3D rotation
$q = (x, y, z, r_x, r_y, r_z)$
$\mathcal{C}$-space is 6D

Configuration space

- A configuration is a **point** in $\mathcal{C}$
- $\mathcal{A}(q)$ is set of **all points** of the robot determined by configuration $q \in \mathcal{C}$
- Therefore, point $q \in \mathcal{C}$ **fully** describes how the robot looks in $\mathcal{W}$
- $\mathcal{C}$ has as many dimensions as robot's DOFs
- $\mathcal{C}$ is considered "high-dimensional" if number of DOFS > 4

Example: a robotic arm with two revolute joints; $q = (\varphi_1, \varphi_1) \rightarrow 2D \quad \mathcal{C}$-space
Robot geometry has two rigid shapes: $\mathcal{A}_1$ and $\mathcal{A}_2$



Obstacles in the configuration space: $\mathcal{C}_{\text{obs}}$

$$\mathcal{C}_{\text{obs}} = \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} \neq \emptyset\}, \quad \mathcal{C}_{\text{obs}} \subseteq \mathcal{C}$$

- $\mathcal{C}_{\text{obs}}$ contains robot-obstacle collisions and self-collisions
- Self-collisions: e.g. in the case of robotic arms
- q is feasible, if it is collision free $\rightarrow q \in \mathcal{C}_{\text{free}}$

$$\mathcal{C}_{\text{free}} = \mathcal{C} \backslash \mathcal{C}_{\text{obs}}$$

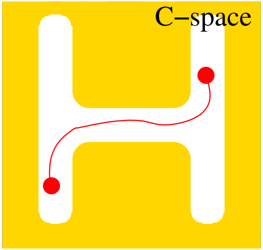
- A **path** in $\mathcal{C}$ is a continuous curve connecting two configurations q_{init} and q_{goal}:

$$\tau : s \in [0, 1] \to \tau(s) \in \mathcal{C}; \quad \tau(0) = q_{\text{init}} \text{ and } \tau(1) = q_{\text{goal}}$$

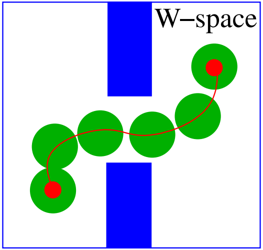
- A **trajectory** is a path parameterized by time

$$\tau : t \in [0, T] \to \tau(t) \in \mathcal{C}$$

- Trajectory/path defines motion in workspace



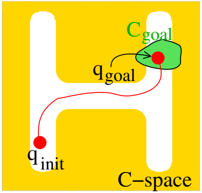
Path in $\mathcal{C}$



Workspace motion

Given

- model of the world $\mathcal{W}$ and robot $\mathcal{A}$
- start $q_{\text{init}} \in \mathcal{C}_{\text{free}}$
- goal region $\mathcal{C}_{\text{goal}} \subseteq \mathcal{C}_{\text{free}}$



Path planning

- To find a collision-free path $\tau(s)$ from q_{init} to $\mathcal{C}_{\text{goal}}$
- i.e., $q(s) \in \mathcal{C}_{\text{free}}$ for all $s \in [0, 1]$, $s(0) = q_{\text{init}}$, $s(1) \in \mathcal{C}_{\text{goal}}$

Motion planning

- To find a collision-free trajectory $\tau(t)$ from q_{init} to $\mathcal{C}_{\text{goal}}$
- i.e., $q(t) \in \mathcal{C}_{\text{free}}$ for all $t \in [0, T]$, $s(0) = q_{\text{init}}$, $s(T) \in \mathcal{C}_{\text{goal}}$

Notes

- The above definition is considered as **feasible path/motion planning**
- Using $\mathcal{C}_{\text{goal}}$ instead of single $q_{\text{goal}} \in \mathcal{C}_{\text{free}}$ is more practical
- No optimality criteria is considered

Completeness and optimality

Completeness

- Algorithm is complete, if for any input it correctly reports in **finite time if there is a solution or no**
- If a solution exists, it **must** return one **in a finite time**
- Computationally very hard (P-Space complete)
- Complete methods exist only for low-dimensional problems

Probabilistic completeness

- Algorithm is prob. complete if for scenarios with existing solution the probability of finding that solution converges to one
- If solution does not exists, the method can run forever

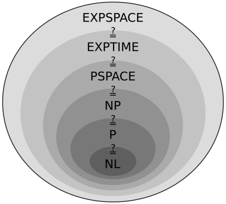
Optimal vs. non-optimal

- Optimal planning: algorithm ensures finding of the optimal solution (according to a criterion)
- Non-optimal: any feasible solution is returned

Complexity of motion planning

Configuration space

- "Converts" planning tasks to a search of path for a **point** in $\mathcal{C}$
- Once we can search $\mathcal{C}$, we can solve any planning problem
- Motion planning is P-Space complete!



Why is planning so difficult?

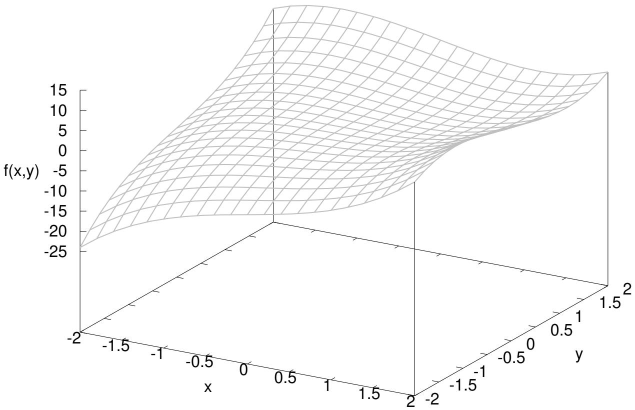
- Because we have to explicitly know $\mathcal{C}$, $\mathcal{C}_{\mathrm{obs}}$ and $\mathcal{C}_{\mathrm{free}}$
- The most important are obstacles $\mathcal{C}_{\mathrm{obs}}$, but they are given implicitly:

$$\mathcal{C}_{\mathrm{obs}} = \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} \neq \emptyset\}, \quad \mathcal{C}_{\mathrm{obs}} \subseteq \mathcal{C}$$

- Implicit definition does not allow to enumerate points in $\mathcal{C}_{\mathrm{obs}}$
- Difficult to determine the nearest colliding configuration

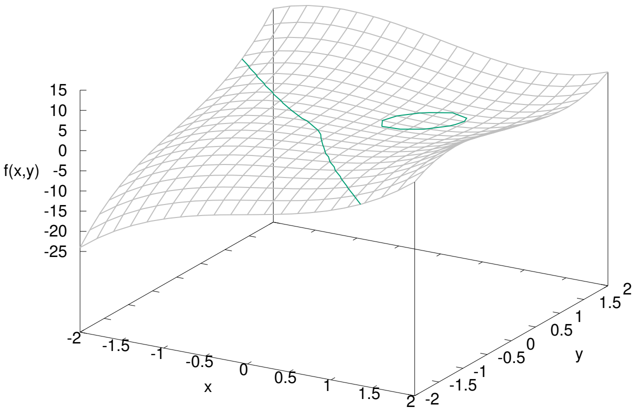
☛ J. Canny. The complexity of robot motion planning. MIT press, 1988.

$$f(x, y) = x^3 - 2xy + y^3$$



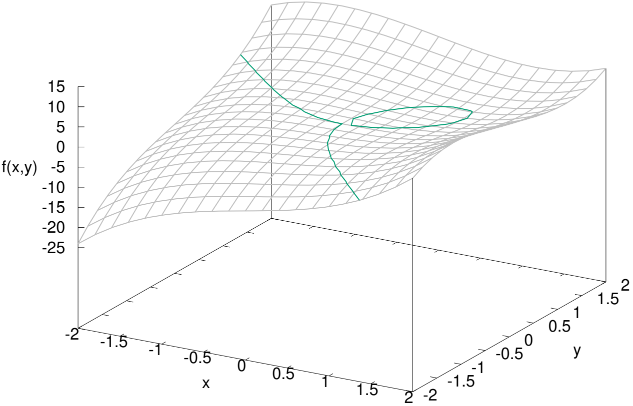
$f(x, y)$

$$f(x, y) = x^3 - 2xy + y^3$$



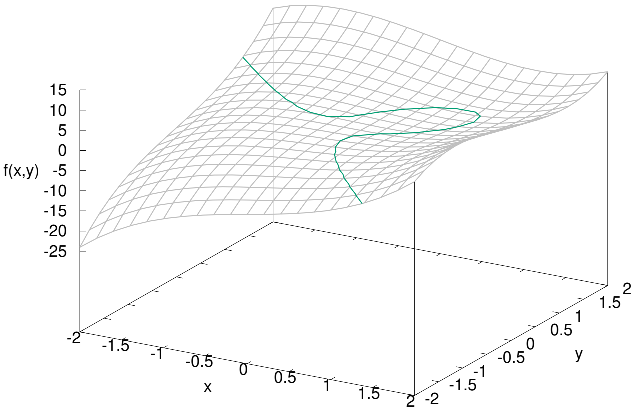
$$f(x, y) = -0.1$$

$$f(x, y) = x^3 - 2xy + y^3$$



$$f(x, y) = 0$$

$$f(x, y) = x^3 - 2xy + y^3$$



$$f(x, y) = 0.1$$

- How to get explicit list of obstacles from the implicit obstacles

$$\mathcal{C}_{\text{obs}} = \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} \neq \emptyset\}, \quad \mathcal{C}_{\text{obs}} \subseteq \mathcal{C}$$

- i.e., how to enumerate points on the border of the obstacles?

Explicit construction of $\mathcal{C}_{\text{obs}}$

- $\mathcal{A}(0)$ is the robot at origin
- $-\mathcal{A}(0)$ is achieved by replacing all $x \in \mathcal{A}(0)$ by $-x$
- Obstacles in $\mathcal{C}$ are determined by the Minkowski sum

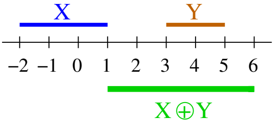
$$\mathcal{C}_{\text{obs}} = \mathcal{O} \oplus -\mathcal{A}(0)$$

Minkowski sum

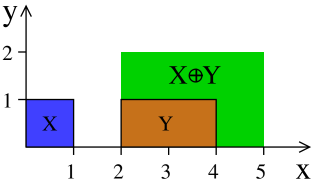
Minkowski sum $\oplus$ of two sets $X, Y \subset R^n$ is

$$X \oplus Y = \{x + y \in \mathbf{R}^n | x \in X \text{ and } y \in Y\}$$

1D example: $X = [-2, 1]$, $Y = [3, 5]$
$X \oplus Y = [1, 6]$



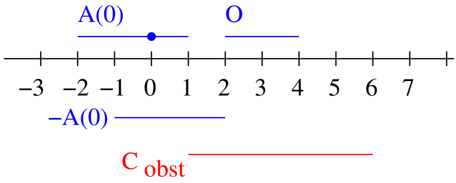
2D example: $X = [0, 1] \times [0, 1]$, $Y = [2, 4] \times [0, 1]$
$X \oplus Y = [2, 5] \times [0, 2]$



Configuration space: 1D case

Example: 1D robot $\mathcal{A} = [-2, 1]$ and obstacle $\mathcal{O} = [2, 4]$:

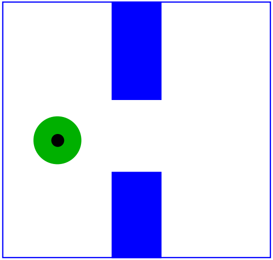
$$\mathcal{C}_{\text{obs}} = \mathcal{O} \oplus -\mathcal{A}(0)$$



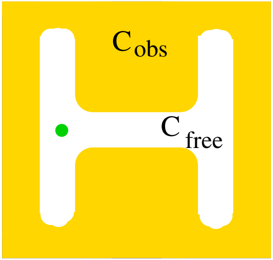
$\mathcal{C}_{\text{obs}} = [1, 6]$

Configuration space: 2D disc robot

- 2D workspace $\mathcal{W} \subseteq \mathbf{R}^2$
- 2D disc robot $\mathcal{A} \subseteq \mathbf{R}^2$, reference point in the disc's center
- We assume **only translation**
- Therefore, configuration $q = (x, y)$ and $\mathcal{C}$ is 2D



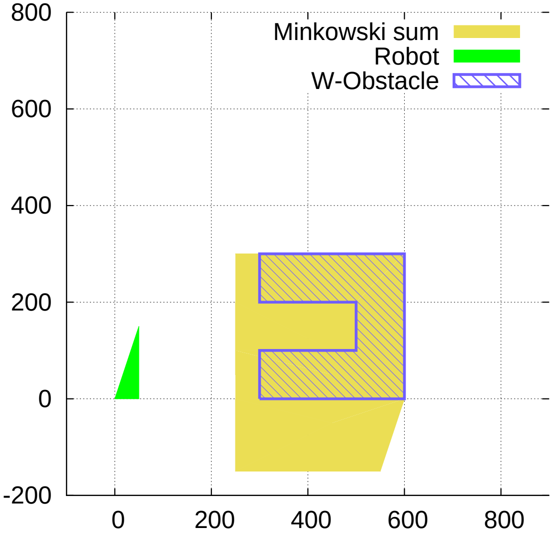
Workspace



Configuration space

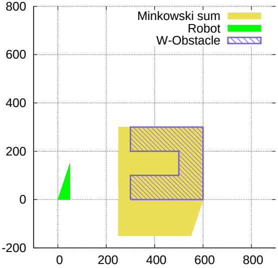
- All $q \in \mathcal{C}_{\text{free}}$ are collision-free $\rightarrow \mathcal{A}(q) \cap \mathcal{O} = \emptyset$
- Volume of $\mathcal{C}_{\text{free}}$ depends both on the robot and obstacles
- What happens if the robot is a point?

- 2D robot, only translation, $q = (x, y) \rightarrow$ 2D $\mathcal{C}$

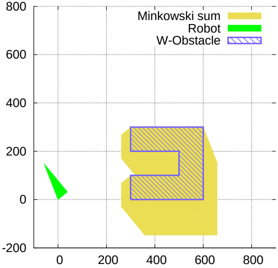


Configuration space: 2D robot II

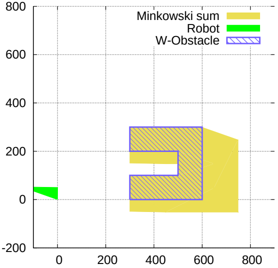
- 2D robot, translation + rotation, $q = (x, y, \varphi) \rightarrow$ 3D $\mathcal{C}$
- Requires to compute Minkowski sum for each rotation



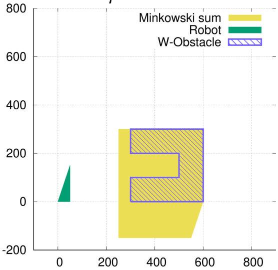
$\varphi = 0$



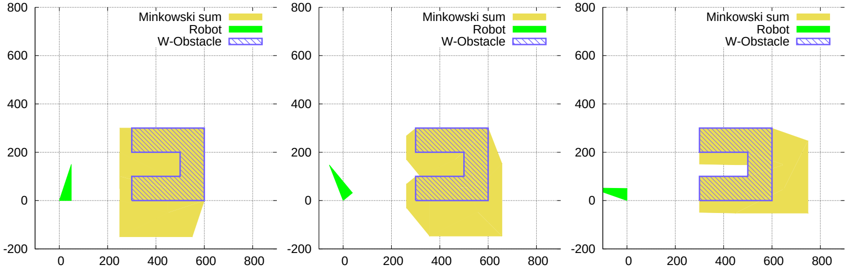
$\varphi = 45$



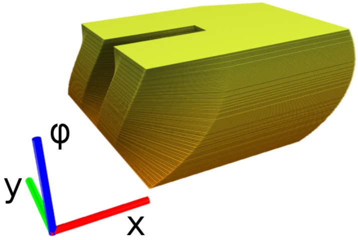
$\varphi = 90$



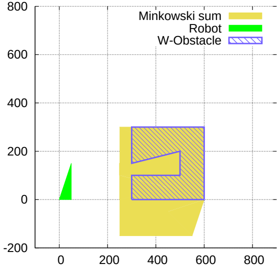
- 2D robot, translation + rotation, $q = (x, y, \varphi) \rightarrow$ 3D $\mathcal{C}$
- Requires to compute Minkowski sum for each rotation



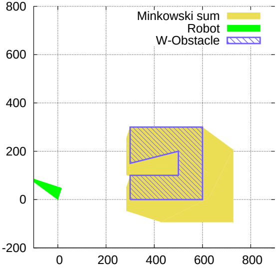
$\varphi = 0$ $\varphi = 45$ $\varphi = 90$



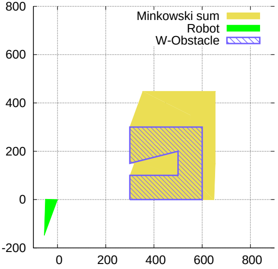
- 2D robot, translation + rotation, $q = (x, y, \varphi) \rightarrow$ 3D $\mathcal{C}$
- Requires to compute Minkowski sum for each rotation



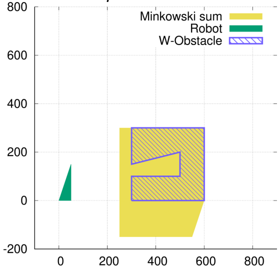
$\varphi = 0$



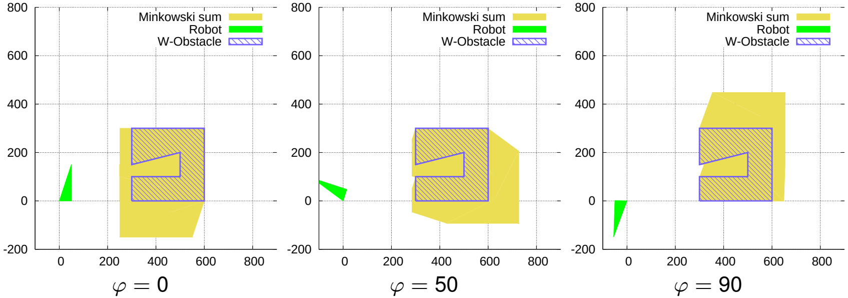
$\varphi = 50$

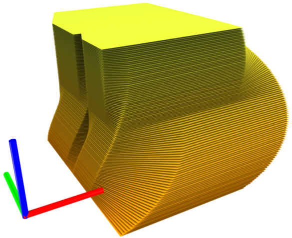


$\varphi = 90$



- 2D robot, translation + rotation, $q = (x, y, \varphi) \to$ 3D $\mathcal{C}$
- Requires to compute Minkowski sum for each rotation





Explicit construction of $\mathcal{C}$

Minkowski sum of two objects of *n* and *m* complexity

2D polygons

- convex $\oplus$ convex, $O(m+n)$
- convex $\oplus$ arbitrary, (mn)
- arbitrary $\oplus$ arbitrary, $(m^2 n^2)$

3D polyhedrons

- convex $\oplus$ convex, $O(mn)$
- arbitrary $\oplus$ arbitrary, $(m^3 n^3)$

- Construction of $\mathcal{C}$ Minkowski sums is straightforward, but ...
- We have only 2D/3D models of robots and obstacles

$\rightarrow$ directly we can construct $\mathcal{C}$ only for "translation only" systems

- Other DOFS need to be discretized and Minkowski sum computed for each combination (!)
- Explicit construction of $\mathcal{C}$ is computationally demanding!
- Not practical for high-dimensional systems
- Explicit construction of $\mathcal{C}_{\text{obs}}$ using Minkowski sum is (generally) too difficult, and it is not practically used.

Motion restrictions

Robots (usually) cannot move arbitrarily

- Kinematic constraints (e.g. 'car-like' vehicle)
- Dynamic constraints (e.g. maximal acceleration)
- Task constraints (e.g 'do not spill the beer')
- These are considered as additional constraints that must be satisfied in path/motion planning

Motion model

- describes how the robot's state changes when input $u \in \mathcal{U}$ is applied at $q \in C$
- $\mathcal{U}$ is a set of all possible inputs

$$\dot{q} = f(q, u)$$

- Discrete version is often used:

$$q_{k+1} = f(q_k, u), \qquad q_{k+1}, q_k \in \mathcal{C}, u \in \mathcal{U}$$

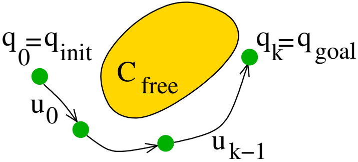
Given

- model of the world $\mathcal{W}$ and robot $\mathcal{A}$, configurations $q_{\mathrm{init}}, q_{\mathrm{goal}} \in \mathcal{C}_{\mathrm{free}}$
- motion model $q' = f(q, u)$ with inputs $\mathcal{U}$

Discrete feasible planning

- Find a finite sequence of actions $\pi_k = (u_0, \ldots, u_{k-1}), u \in \mathcal{U}$ such that

$$
\begin{aligned}
q_{k+1} &= f(q_k, u_k) \\
q_0 &= q_{\mathrm{init}} \\
q_k &= q_{\mathrm{goal}} \\
q_k &\in \mathcal{C}_{\mathrm{free}}
\end{aligned}
$$



- The sequence of states $(q_1, \ldots, q_k)$ can be derived from the motion model starting from q_0 and applying $q_{k+1} = f(q_k, u_k)$ subsequently
- Is this plan optimal?

Discrete optimal planning

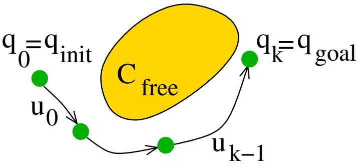
- Let $L(\pi_k)$ is the cost of the sequence $\pi_k = (u_0, \ldots, u_{k-1})$

$$L(\pi_k) = l_f(q_k) + \sum_{i=0}^{k-1} l(q_i, u_i)$$

- the final term $l_f(q_k) = 0$ if $q_k = q_{\mathrm{goal}}$; it is ∞ otherwise

Discrete optimal planning

$$\begin{aligned} \underset{\pi_k=(u_0,\ldots,u_{k-1})}{\text{minimize}} \quad & L(\pi_k) \\ \text{subject to} \quad & q_{k+1} = f(q_k, u_k) \\ & q_0 = q_{\mathrm{init}} \\ & q_k = q_{\mathrm{goal}} \\ & q_k \in \mathcal{C}_{\mathrm{free}} \end{aligned}$$



- $L(\pi_k) = \infty$ means infeasible solution
- $L(\pi_k) < \infty$ means a feasible solution with the cost $L(\pi_k)$

Discrete optimal control

- Optimal control for a discrete-time (and finite horizon)
- initial state is x_i, goal state x_n may be given (or not)

$$\underset{u_i,\ldots,u_{N-1},(x_i),\ldots,x_n}{\text{minimize}} \quad \left(\phi(x_n, N) + \sum_{k=i}^{N-1} L_k(x_k, u_k)\right)$$

$$\text{subject to} \quad \begin{aligned} x_{k+1} &= f_k(x_k, u_k) \\ u_{lb} \leq u_k &\leq u_{ub} \\ x_{lb} \leq x_k &\leq x_{ub} \end{aligned}$$

Discrete optimal control (generally)

$$\underset{x\in\mathbf{R}^{n(N-i)},u\in R^{m(N-i)}}{\text{minimize}} \quad J(x, u)$$

$$\text{subject to} \quad \begin{aligned} g(x, u) &= 0 \\ h(x, u) &\leq 0 \end{aligned}$$

equations by Z. Hurak: Discrete-time optimal control — direct approach (lectures notes of ORR)

Motion planning vs control

- Optimal control and optimal (path/motion) planning are (generally) the same
- Both can find path/trajectory from start to goal
- What is the practical difference?

Path planning

- Solution is achieved by searching $\mathcal{C}$-space
- Can work with explicit (combinatorial planning) or implicit obstacles (sampling-based planning)
- Difficult to react on changes (robot control error, dynamic obstacles) $\rightarrow$ replanning
- Replanning requires to solve the problem from scratch $\rightarrow$ slow

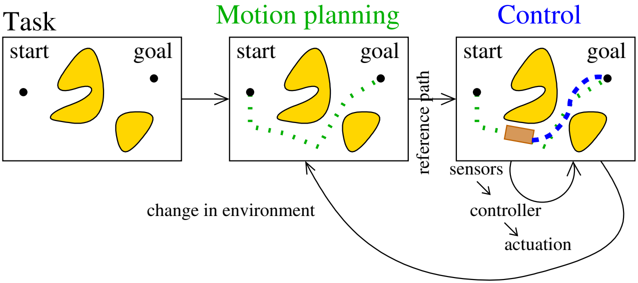
Motion planning vs control

- Optimal control and optimal (path/motion) planning are (generally) the same
- Both can find path/trajectory from start to goal
- What is the practical difference?

Control

- Trajectory is achieved via mathematical optimization
  - we (typically) need “a gradient” $\rightarrow$, e.g. ‘distance to the nearest obstacle’, its derivative etc.
  - this requires an explicit representation of $\mathcal{C}$ resp. $\mathcal{C}_{\mathrm{obs}}$
- Difficult to find first (feasible) solution $\rightarrow$ large search space
- Suitable for following reference, e.g. reference trajectory from motion planning

Planning + control



- Global plan delivered by motion planning
- Sensing (actual position, speed, etc.) controlled along planned path
- i.e., errors in actuation are handled by control
- Replanning when global change occurs (e.g. new obstacle that cannot be handled by control)

Does not make sense to solve motion plan by control-theory methods

Does not make sense to control via planning!

Confusion in terminology

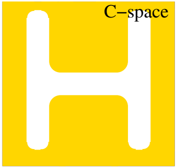
- Path/motion planning are studied in several disciplines
  - Robotics, computation geometry, mathematics, biology
  - . . . since 1950's !
- Each field uses different meaning for "path" and "trajectory"

  . . . and different meaning for path/motion planning
- this continues up to now

What is a "trajectory"?

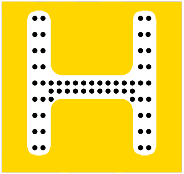
- Robotics (including this lecture): path + time
- Control-oriented part of robotics: path + time + control inputs
- Computational biology: 3D path of atom(s) (with or without time)

Before you start to solve a planning problem, define (or agree on) the basic terms first!



Continuous space

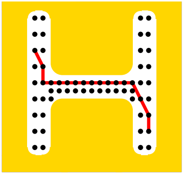
↓



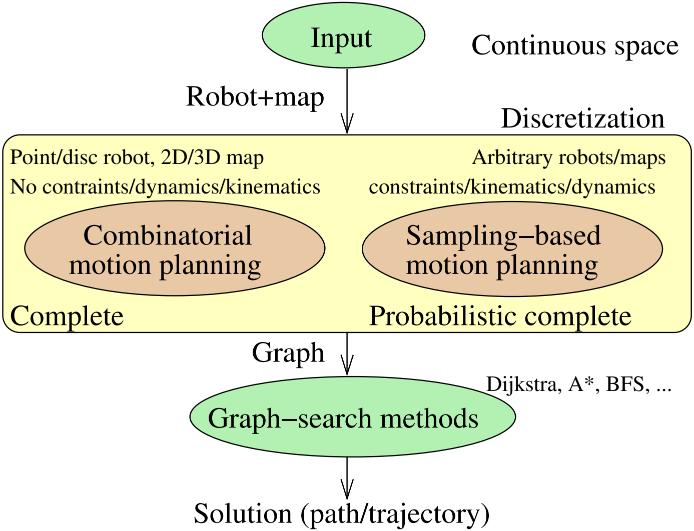
Discretization

↓

Search



The art of motion planning



Summary of the lecture

- Motion planning: how to move objects and avoid obstacles
- Configuration space $\mathcal{C}$
- Generally, planning leads to search in continuous $\mathcal{C}$
- But we (generally) don't have explicit representation of $\mathcal{C}$
- We have to first create a discrete representation of $\mathcal{C}$
- and search it by graph-search methods